Collective Rationality in Graph Aggregation

Ulle Endriss
Institute for Logic, Language and Computation
University of Amsterdam

[ joint work with Umberto Grandi ]
Talk Outline

• Graph Aggregation
• Collective Rationality wrt. a Graph Property
• A General Impossibility Result
Graph Aggregation

Fix a finite set of vertices $V$. A (directed) graph $G = \langle V, E \rangle$ based on $V$ is defined by a set of edges $E \subseteq V \times V$ (thus: graph = edge-set).

Everyone in a finite group of agents $\mathcal{N} = \{1, \ldots, n\}$ provides a graph, giving rise to a profile $E = (E_1, \ldots, E_n)$.

An aggregator is a function mapping profiles to collective graphs:

$$F : (2^{V \times V})^n \to 2^{V \times V}$$

Example: majority rule (accept an edge iff $\frac{n}{2}$ of the individuals do)
Axioms

We may want to impose certain axioms on $F : (2^{V \times V})^n \rightarrow 2^{V \times V}$, e.g.:

- **Anonymous**: $F(E_1, \ldots, E_n) = F(E_{\sigma(1)}, \ldots, E_{\sigma(n)})$
- **Nondictatorial**: for no $i^* \in \mathcal{N}$ you always get $F(E) = E_{i^*}$
- **Unanimous**: $F(E) \supseteq E_1 \cap \cdots \cap E_n$
- **Grounded**: $F(E) \subseteq E_1 \cup \cdots \cup E_n$
- **Neutral**: $N^E_e = N^E_{e'}$ implies $e \in F(E) \iff e' \in F(E)$
- **Independent**: $N^E_{e} = N^{E'}_{e}$ implies $e \in F(E) \iff e \in F(E')$

For technical reasons, we’ll restrict some axioms to nonreflexive edges $(x, y) \in V \times V$ with $x \neq y$ (NR-neutral, NR-nondictatorial).

**Notation**: $N^E_e = \{i \in \mathcal{N} \mid e \in E_i\} = \text{coalition}$ accepting edge $e$ in $E$
Collective Rationality

Aggregator $F$ is collectively rational (CR) for graph property $P$ if, whenever all individual graphs $E_i$ satisfy $P$, so does the outcome $F(E)$.

Examples for graph properties: reflexivity, transitivity, seriality, . . .
Example

Three agents each provide a graph on the same set of four vertices:

If we aggregate using the \textit{majority rule}, we obtain this graph:

\textbf{Observations:}

\begin{itemize}
  \item Majority rule not collectively rational for \textit{seriality}.
  \item But \textit{symmetry} is preserved.
  \item So is \textit{reflexivity} (easy: individuals violate it).
\end{itemize}
A Simple Possibility Result

Proposition 1 Any unanimous aggregator is CR for reflexivity.

Proof: If every individual graph includes edge \((x, x)\), then unanimity ensures the same for the collective outcome graph. ✓
Arrow’s Theorem

Our formulation in graph aggregation:

For $|V| \geq 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for reflexivity, transitivity and completeness.

This implies the standard formulation, because:

- weak preference orders = reflexive, transitive, complete graphs
- (weak) Pareto + CR $\Rightarrow$ unanimous + grounded
- nondictatorial = NR-nondictatorial for reflexive graphs
- CR for reflexivity is vacuous (implied by unanimity)

We wanted to know:

- For what other classes of graphs does this go through?
Our General Impossibility Theorem

Our main result:

For $|V| \geq 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for any graph property that is contagious, implicative and disjunctive.

where:

- Implicative $\approx [\wedge S^+ \land \neg \lor S^-] \rightarrow [e_1 \land e_2 \rightarrow e_3]$
- Disjunctive $\approx [\wedge S^+ \land \neg \lor S^-] \rightarrow [e_1 \lor e_2]$
- Contagious $\approx$ for every accepted edge, there are some conditions under which also one of its “neighbouring” edges is accepted

Examples:

- *Transitivity* is contagious and implicative
- *Completeness* is disjunctive
- *Connectedness* $[xEy \land xEz \rightarrow (yEz \lor zEy)]$ has all 3 properties
Last Slide

We have introduced *graph aggregation* as a generalisation of preference aggregation and then considered *collective rationality*.

Why is this interesting?

- Potential for *applications*: abstract argumentation, social networks
- Deep insights into the *structure of impossibilities*: direct link between CR requirements and neutrality/ultrafilter conditions