Collective Rationality in Graph Aggregation

Ulle Endriss
Institute for Logic, Language and Computation
University of Amsterdam

[joint work with Umberto Grandi]

Talk Outline

- Graph Aggregation
- Collective Rationality wrt. a Graph Property
- A General Impossibility Result

Graph Aggregation

Fix a finite set of *vertices* V. A (directed) *graph* $G = \langle V, E \rangle$ based on V is defined by a set of *edges* $E \subseteq V \times V$ (thus: graph = edge-set).

Everyone in a finite group of *agents* $\mathcal{N} = \{1, \ldots, n\}$ provides a graph, giving rise to a *profile* $\boldsymbol{E} = (E_1, \ldots, E_n)$.

An *aggregator* is a function mapping profiles to collective graphs:

$$F : (2^{V \times V})^n \to 2^{V \times V}$$

Example: *majority rule* (accept an edge *iff* $> \frac{n}{2}$ of the individuals do)

Axioms

We may want to impose certain *axioms* on $F : (2^{V \times V})^n \to 2^{V \times V}$, e.g.:

- *Anonymous:* $F(E_1, \ldots, E_n) = F(E_{\sigma(1)}, \ldots, E_{\sigma(n)})$
- *Nondictatorial:* for no $i^\star \in \mathcal{N}$ you always get $F(\boldsymbol{E}) = E_{i^\star}$
- *Unanimous:* $F(\boldsymbol{E}) \supseteq E_1 \cap \cdots \cap E_n$
- *Grounded:* $F(\boldsymbol{E}) \subseteq E_1 \cup \cdots \cup E_n$
- *Neutral:* $N_e^{\boldsymbol{E}} = N_{e'}^{\boldsymbol{E}}$ implies $e \in F(\boldsymbol{E}) \Leftrightarrow e' \in F(\boldsymbol{E})$
- *Independent:* $N_e^{\boldsymbol{E}} = N_e^{\boldsymbol{E'}}$ implies $e \in F(\boldsymbol{E}) \Leftrightarrow e \in F(\boldsymbol{E'})$

For technical reasons, we'll restrict some axioms to *nonreflexive edges* $(x, y) \in V \times V$ with $x \neq y$ (NR-neutral, NR-nondictatorial).

Notation: $N_e^{\boldsymbol{E}} = \{i \in \mathcal{N} \mid e \in E_i\}$ = *coalition* accepting edge e in $\boldsymbol{E}$

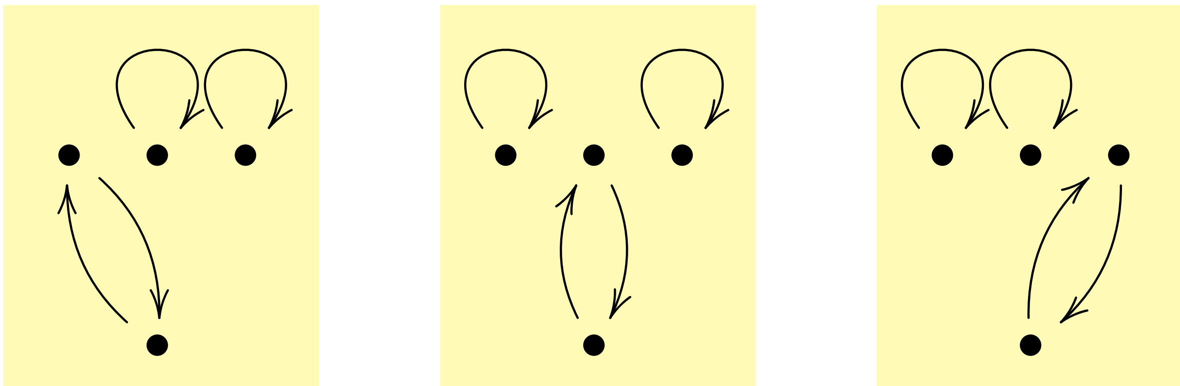
Collective Rationality

Aggregator F is *collectively rational* (CR) for graph property P if, whenever all individual graphs E_i satisfy P, so does the outcome $F(\boldsymbol{E})$.

Examples for *graph properties:* reflexivity, transitivity, seriality, ...

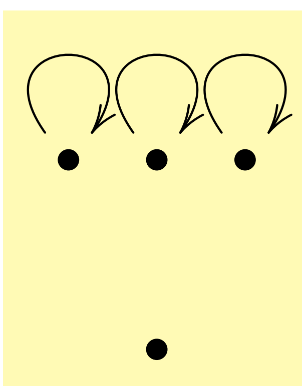
Example

Three agents each provide a graph on the same set of four vertices:



1 2 3

If we aggregate using the *majority rule*, we obtain this graph:



Observations:

- Majority rule not collectively rational for *seriality*.
- But *symmetry* is preserved.
- So is *reflexivity* (easy: individuals violate it).

A Simple Possibility Result

Proposition 1 *Any unanimous aggregator is CR for reflexivity.*

Proof: If every individual graph includes edge (x, x), then unanimity ensures the same for the collective outcome graph. ✓

Arrow's Theorem

Our formulation in graph aggregation:

> *For $|V| \geqslant 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for ~~reflexivity~~, transitivity and completeness.*

This implies the standard formulation, because:

- weak preference orders = reflexive, transitive, complete graphs
- (weak) Pareto + CR $\Rightarrow$ unanimous + grounded
- nondictatorial = NR-nondictatorial for reflexive graphs
- CR for reflexivity is vacuous (implied by unanimity)

We wanted to know:

- ▶ For what other classes of graphs does this go through?

Our General Impossibility Theorem

Our main result:

> *For $|V| \geqslant 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for any graph property that is contagious, implicative and disjunctive.*

where:

- Implicative $\approx [\bigwedge S^+ \wedge \neg \bigvee S^-] \to [e_1 \wedge e_2 \to e_3]$
- Disjunctive $\approx [\bigwedge S^+ \wedge \neg \bigvee S^-] \to [e_1 \vee e_2]$
- Contagious $\approx$ for every accepted edge, there are some conditions under which also one of its "neighbouring" edges is accepted

Examples:

- *Transitivity* is contagious and implicative
- *Completeness* is disjunctive

$\Big\} \Rightarrow$ *Arrow's Theorem*

- *Connectedness* $[xEy \wedge xEz \to (yEz \vee zEy)]$ has all 3 properties

Last Slide

We have introduced *graph aggregation* as a generalisation of preference aggregation and then considered *collective rationality*.

Why is this interesting?

- Potential for *applications:* abstract argumentation, social networks
- Deep insights into the *structure of impossibilities:* direct link between CR requirements and neutrality/ultrafilter conditions