Halfway between Points and Intervals: A Temporal Logic Based on Ordered Trees

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Talk Overview

• Halfway between Points and Intervals
  finding a compromise between complexity and expressiveness

• Modal Logics of Ordered Trees (OTL)
  syntax and semantics; examples

• Interpreting OTL as a Temporal Interval Logic
  past and future; properties and events

• Technical Results
  axiomatisation and decidability

• Conclusion
  recap and discussion of future work
Motivation

• For many applications, time intervals seem more appropriate than time points, but modal interval logics (such as Halpern and Shoham’s logic) tend to be undecidable.

• Points (and intervals) can be used to model that one event takes place before (or after) another event.

• Intervals are more expressive than time points:
  – two intervals may overlap
  – an interval may be decomposed into several others

• As a compromise, we study a temporal logic where primitive objects are “halfway” between points and intervals: they can be decomposed into smaller units (like time intervals) but they cannot overlap (like time points).
Modal Logics of Ordered Trees (OTL)

Models are based on ordered trees. Besides the usual propositional connectives we have a number of modal operators, for example:

\[
\begin{align*}
\Theta \varphi & \quad \text{“} \varphi \text{ is true at the next righthand sibling (if any)} \text{”} \\
\diamond \varphi & \quad \text{“} \varphi \text{ is true at some righthand sibling} \text{”} \\
\circ \varphi & \quad \text{“} \varphi \text{ is true at the parent (if any)} \text{”} \\
\psi \varphi & \quad \text{“} \varphi \text{ is true at some ancestor} \text{”} \\
\diamond \varphi & \quad \text{“} \varphi \text{ is true at some child} \text{”} \\
\diamond \varphi & \quad \text{“} \varphi \text{ is true at some descendant} \text{”}
\end{align*}
\]

We also have modalities (Θ and ◊) to refer to lefthand siblings. All corresponding box-operators are definable, for example:

\[\Box \varphi = \neg \diamond \neg \varphi \quad \text{“} \varphi \text{ is true at all righthand siblings} \text{”}\]
Examples

• A node $t$ is the root of the tree iff $\text{ROOT}$ is true at $t$:

$$\text{ROOT} = \Box \bot$$

• Similar formulas identify leftmost and rightmost siblings:

$$\text{LEFTMOST} = \Box \bot \quad \text{RIGHTMOST} = \Box \bot$$

• An ordered tree $\mathcal{T}$ is a binary tree iff $\text{BINARY}$ is valid in $\mathcal{T}$ (or, equivalently, iff $\text{BINARY}$ is globally true in a model based on $\mathcal{T}$):

$$\text{BINARY} = (\text{LEFTMOST} \leftrightarrow \text{RIGHTMOST}) \rightarrow \text{ROOT}$$

• An ordered tree $\mathcal{T}$ is “discretely branching” iff the formula $\text{DISCRETE}$ is valid in $\mathcal{T}$:

$$\text{DISCRETE} = \Box (\exists A \rightarrow A) \rightarrow (\Diamond \Box A \rightarrow \Box A)$$

This is the standard axiom schema familiar from temporal logic to characterise discrete flows of time.
OTL as a Temporal Interval Logic

We can interpret OTL as a (restricted) interval logic as follows:

- nodes in a tree represent time intervals;
- descendants represent subintervals; and
- the order declared over siblings represents an earlier-later ordering over time intervals.

But this raises some questions:

- What is the meaning of, say, the $\Diamond$-operator? Is it a proper future modality?
- Are models where, say, $\varphi$ is true at some node $t$ but not at all of $t$’s children meaningful under this temporal interpretation?
Past and Future

If you are allowed to move up before and down after moving to the right, you can reach all the nodes to the right.

Let $\Diamond^* \varphi = \varphi \lor \Diamond \varphi$ and $\Diamond^* \varphi = \varphi \lor \Diamond^+ \varphi$.

We can now define a global future modality as follows:

$$\Diamond \varphi = \Diamond^* \Diamond \Diamond^* \varphi$$
halfway between Points and Intervals

Ontological Considerations

Consider the following two basic propositions:

(1) The sun is shining.

(2) I move the pen from the table onto the OHP.

Propositions like (1) are sometimes called properties; propositions like (2) are sometimes called events (Allen, 1984).
Properties

Properties like “The sun is shining.” are homogeneous propositions (Shoham, 1987), which we can capture in OTL as follows:

\[
\begin{align*}
\text{DOWNWARD-HEREDITARY}(\varphi) &= \varphi \rightarrow \Box^+ \varphi \\
\text{UPWARD-HEREDITARY}(\varphi) &= \lozenge \top \rightarrow (\Box^+ \varphi \rightarrow \varphi) \\
\text{HOMOGENEOUS}(\varphi) &= \text{DOWNWARD-HEREDITARY}(\varphi) \land \\
&\quad \text{UPWARD-HEREDITARY}(\varphi)
\end{align*}
\]

Then $\varphi$ is a homogeneous proposition (with respect to a given model $\mathcal{M}$) iff $\text{HOMOGENEOUS}(\varphi)$ is globally true in $\mathcal{M}$. 
Technical Results

• A complete *axiomatisation* of the fragment of OTL excluding the transitive descendant operator \((\lozenge^+)\) is available.
  – The most interesting axioms are:
    
    (X1) \(\lozenge A \rightarrow \circlearrowleft \lozenge A\)
    
    (X2) \(\circlearrowleft \lozenge A \rightarrow (A \lor \lozenge A \lor \lozenge A)\)
  – Problems with proving completeness for the full logic: transitive closure + irreflexivity + interaction

• OTL is *decidable*.
  – Proof largely builds on a reduction to Rabin’s Theorem.
  – Note that 2-dimensional modal logics with interacting modalities (such as products) are often undecidable.
    
    (1) \(\lozenge^* \lozenge^* A \rightarrow \lozenge^* \lozenge^* A\) (right-commutativity)
    
    (2) \(\lozenge^* \square^* A \rightarrow \square^* \lozenge^* A\) (Church-Rosser property)
Conclusion

- We have introduced a simple yet expressive modal logic for talking about ordered trees.
- Original motivation: linear temporal logic + zoom
- Compromise between point- and interval-based temporal logics:
  - can model subintervals, but not overlapping intervals
  - decidable (unlike many interval logics)
- Future work:
  - prove decidability directly (rather than by reduction to S2S)
  - give a complete axiomatisation for the full logic
  - extend results to cover until-style operators
  - develop a decision procedure (possibly Tableau-based)
  - use OTL to represent (nested) communication protocols