Preference Aggregation with Restricted Ballot Languages: Sincerity and Strategy-Proofness

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Problem

Two common assumptions in voting theory:

- Voters have *preferences* that are *total orders* over candidates.
- Voters vote by submitting a structure just like their preferences, truthfully or not (*ballots* and preferences have *the same* structure).

But this is sometimes inappropriate:

- For lack of information or processing resources, voters may be *unable to rank* all candidates (in their mind or on the ballot sheet).
- To reduce complexity of communication, we may want to design voting rules that work with ballots of *bounded size*.
- For *approval voting*, ballots cannot be encoded using total orders.

Talk Outline

- Our model: preferences and ballots can be different structures
- Sincerity:
  - Important notion of truthfulness can become meaningless
  - Replace it with sincerity: as truthful as possible
  - Three possible definitions compared
- Strategy-proofness:
  - Definition of strategy-proofness in terms of sincerity
  - Two positive results: some rules are strategy-proof
  - Computational considerations
- Conclusion

Our Model

Preferences $\mathcal{P}$ could be any set of

- *preorders* (reflexive and transitive relations) over $\mathcal{C}$, i.e., allowing for *strict rankings*, *indifferences*, and *incomparabilities*;
- including *partial* (no indifferences), *weak* (no incomparabilities) and *total orders* (only strict rankings).

The ballot language $\mathcal{B}$ could also be any set of

- preorders — except that a ballot should not force a particular strict ranking on any given pair of candidates.

In the standard model, $\mathcal{P} = \mathcal{B} =$ all total orders over $\mathcal{C}$.

A *voting procedure* is a function $f : \mathcal{B}^n \rightarrow 2^{\mathcal{C}}$.

Sincerity

Problem: Given a ballot language $\mathcal{B}$ and a true preference relation p, voting *truthfully* may be *impossible* in this model (if $p \notin \mathcal{B}$).

Question: What are the *sincere* ballots $b \in \mathcal{B}$ wrt. p?

Three possible definitions:

- Ballot $b \in \mathcal{B}$ is *minimally sincere* wrt. p [$b \in \textsc{Sin}_{\mathcal{B}}^{\mathrm{min}}(p)$] if b and p do not *strictly* rank two candidates in opposite ways.
- Ballot $b \in \mathcal{B}$ is *qualitatively sincere* wrt. p [$b \in \textsc{Sin}_{\mathcal{B}}^{\mathrm{qual}}(p)$] if agreement between b and p is maximal wrt. *set-inclusion*.
- Ballot $b \in \mathcal{B}$ is *quantitatively sincere* wrt. p [$b \in \textsc{Sin}_{\mathcal{B}}^{\mathrm{quan}}(p)$] if agreement between b and p is maximal wrt. *cardinality*.

Example

Suppose your *true* preferences are $\mathtt{A} \succ \mathtt{B} \succ \mathtt{C} \succ \mathtt{D}$.

5 of the 15 syntactically valid *approval ballots:*

```
(1)   A       (2)  A B     (3)  A B C    (4)  A B C D    (5)  A C
      |              |            |                              |
    B C D           C D           D                             B D
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According to our definitions —

- Ballots (1)–(4) are *minimally sincere*.
  This corresponds to the standard notion of sincerity for AV.
- Ballots (1)–(3) are *qualitatively sincere*.
  As above, but now excluding the abstention ballot.
- Only ballot (2) is *quantitatively sincere* (most agreements).

Properties

- There is a natural ordering over our notions of sincerity, and it is always *possible* to be sincere:

  Theorem 1 *Let p be a preorder and let $\mathcal{B}$ be a ballot language. Then $\textsc{Sin}^{\text{min}}_{\mathcal{B}}(p) \supseteq \textsc{Sin}^{\text{qual}}_{\mathcal{B}}(p) \supseteq \textsc{Sin}^{\text{quan}}_{\mathcal{B}}(p) \supset \emptyset$.*

- If you *can* be truthful, then this is the *only* way to be sincere:

  Theorem 2 *If $\mathcal{B} \supseteq \mathcal{P}$, then $\textsc{Sin}^{\text{qual}}_{\mathcal{B}}(p) = \textsc{Sin}^{\text{quan}}_{\mathcal{B}}(p) = \{p\}$ for all $p \in \mathcal{P}$. (Does not apply to minimal sincerity though.)*

- The three notions *coincide* for the standard form of balloting:

  Theorem 3 *If $\mathcal{B}$ is the set of all total orders, then we have $\textsc{Sin}^{\text{min}}_{\mathcal{B}}(p) = \textsc{Sin}^{\text{qual}}_{\mathcal{B}}(p) = \textsc{Sin}^{\text{quan}}_{\mathcal{B}}(p)$ for all preorders p.*

Lifting Preferences

Goal: we want to define a voting procedure as strategy-proof if it never gives voters an *incentive* to not cast a sincere ballot . . .

But: a voting procedure can have more than one winner. Hence, when voters strategise, they do so with respect to *sets of winners*. So we need to *lift their preferences* from candidates to sets of candidates.

Example: the *Gärdenfors axioms* define a partial order $\lhd_p$ on $2^{\mathcal{C}} \setminus \{\emptyset\}$ (nonempty sets of candidates) given a preorder p on $\mathcal{C}$ (candidates).

- $S \cup \{x\} \lhd_p S$ whenever $x \prec_p y$ for all $y \in S$
- $S \lhd_p S \cup \{y\}$ whenever $x \prec_p y$ for all $x \in S$

Generalised Strategy-Proofness

Fix possible preferences $\mathcal{P}$ and ballot language $\mathcal{B}$.

Fix notion of sincerity $\textsc{Sin}_{\mathcal{B}} : \mathcal{P} \to 2^{\mathcal{B}}$ and lifting $\lhd_p$ for all $p \in \mathcal{P}$.

- A voting procedure $f : \mathcal{B}^n \to 2^{\mathcal{C}}$ is *g-strategy-proof* if, for all voters i with *true preference* $p_i \in \mathcal{P}$ and for all ballot vectors $b \in \mathcal{B}^n$, there exists a sincere ballot $b'_i \in \textsc{Sin}_{\mathcal{B}}(p_i)$ such that $f(b_{-i}, b'_i) \not\lhd_{p_i} f(b)$.

Results

For all results, we assume that the Gärdenfors lifting $\lhd_p$ is used.

Theorem 4 *Approval voting is g-strategy-proof wrt. qualitative (and minimal, but not quantitative) sincerity (for total order preferences).*

Theorem 5 *For 2-level preferences, all of plurality, Borda, and approval voting are g-strategy-proof wrt. quantitative sincerity.*

The latter generalises to a wide range of procedures (*"longest-path voting with neutral ballot languages"*), at least for minimal sincerity.

Computational Complexity

How hard is it to be sincere? Degrees of g-strategy-proofness:

- *Blind g-strategy-proofness:* can play optimally and sincerely without requiring any information about other ballots — $O(1)$

  Example: plurality with just two candidates

- *Tractable g-strategy-proofness:* need to know ballots (or similar), but can compute a sincere optimal ballot in polynomial time

  Example: Borda for 2-level preferences (*theorem in paper*)

- *Intractable g-strategy-proofness:* need to know ballots (or similar) and finding a sincere optimal ballot is computationally intractable

  (No known examples.)

Conclusion

- Dropping assumption that preferences are total orders and ballots are just reported preferences leads to an interesting model.
- Proposed generalised definition of strategy-proofness and showed that Gibbard-Satterthwaite-like theorems are less prevalent here.
- Also: some results on comparing different notions of sincerity + starting point for complexity-theoretic investigations of the model.