

# Representation Matters: Characterisation and Impossibility Results for Interval Aggregation

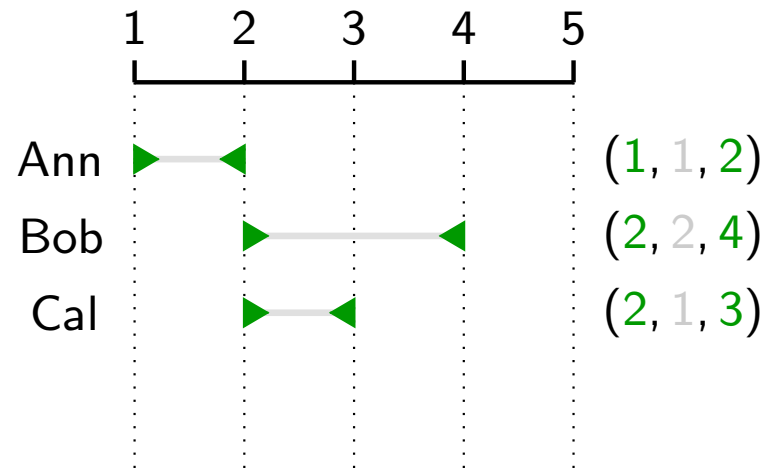
Ulle Endriss

Institute for Logic, Language and Computation

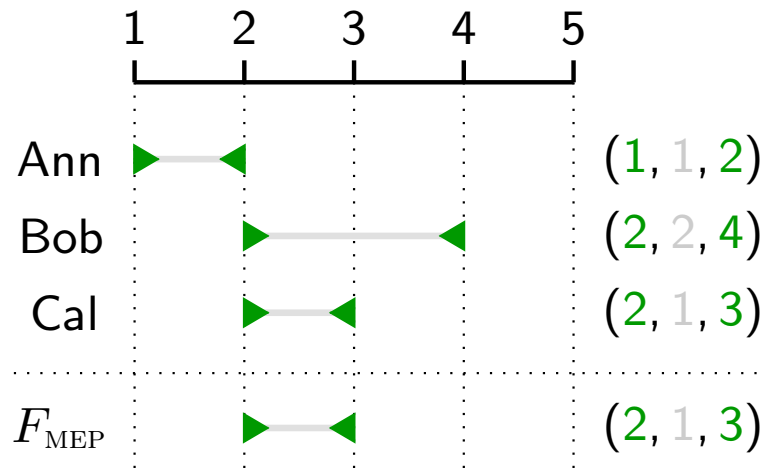
University of Amsterdam

[ joint work with Arianna Novaro and Zoi Terzopoulou ]

## Example

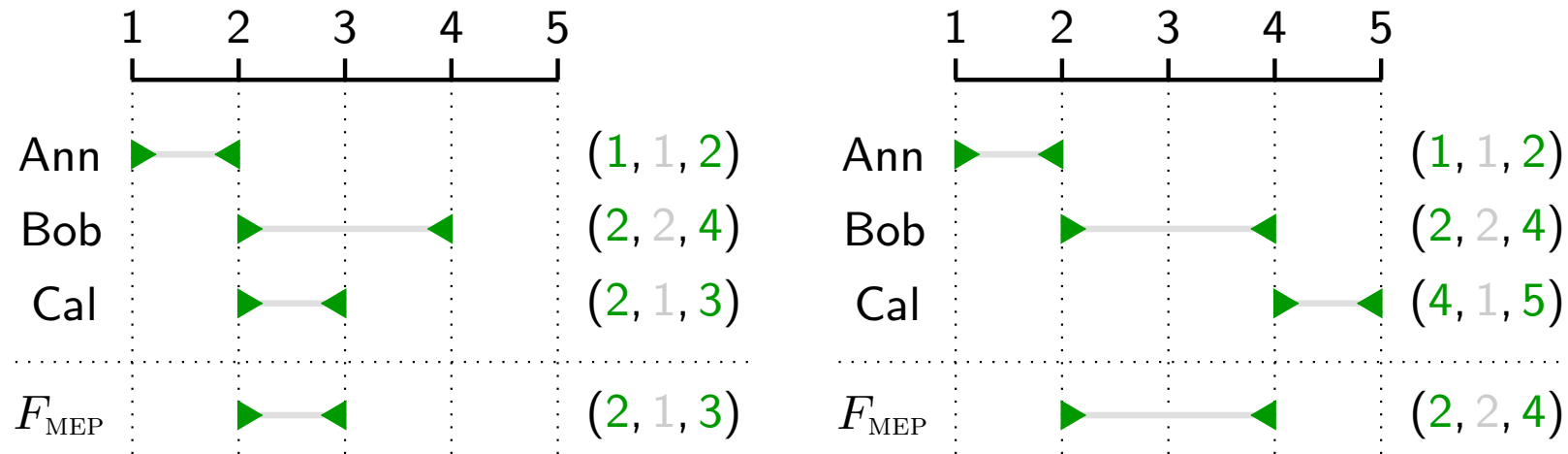


## Example



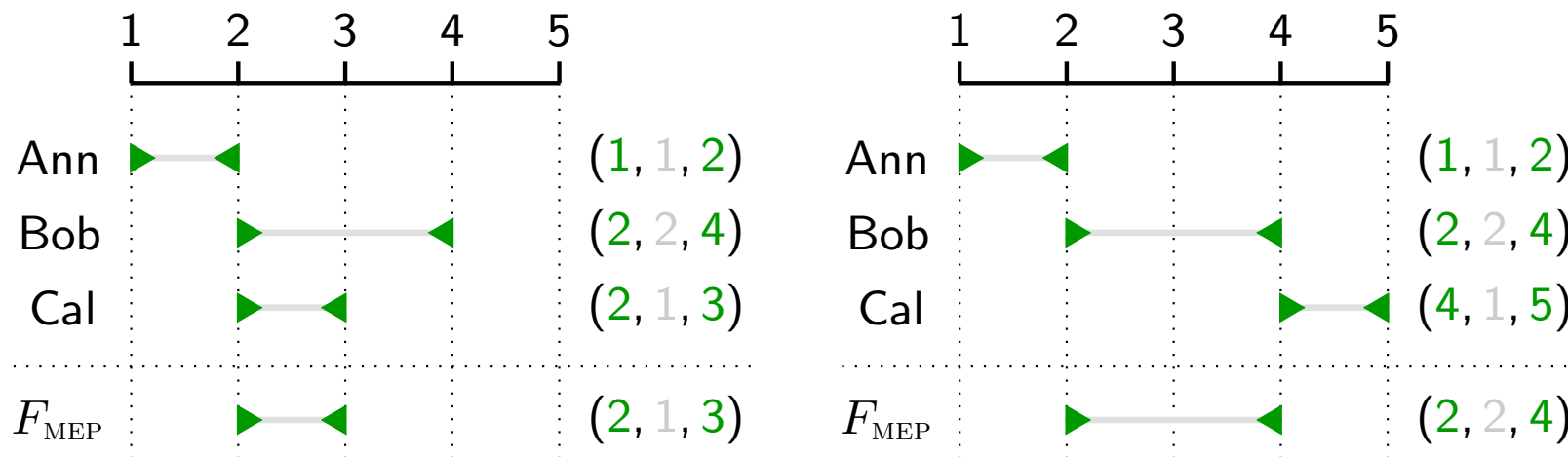
Let's use the *median-endpoint rule* to aggregate!

## Example



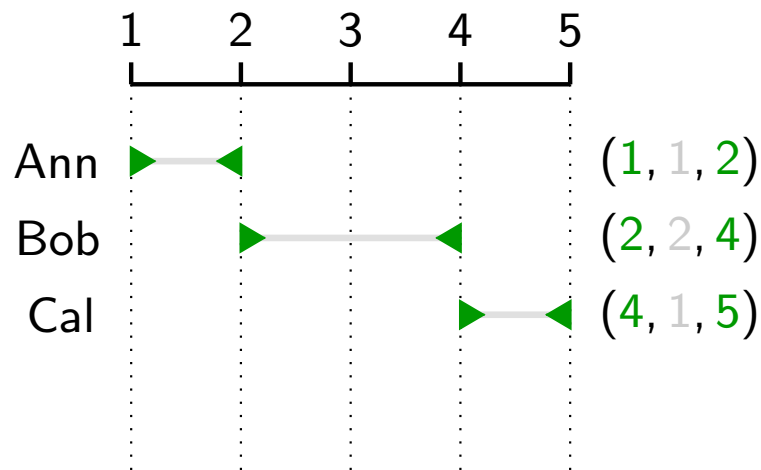
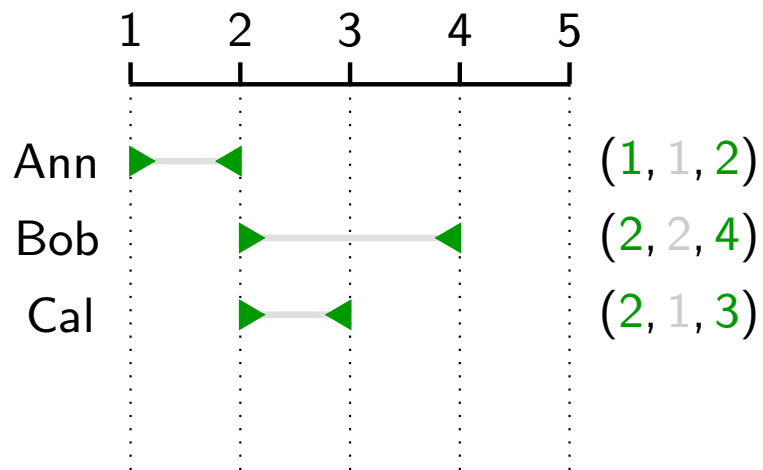
*Let's use the **median-endpoint rule** to aggregate!*  
(now for both scenarios)

## Example



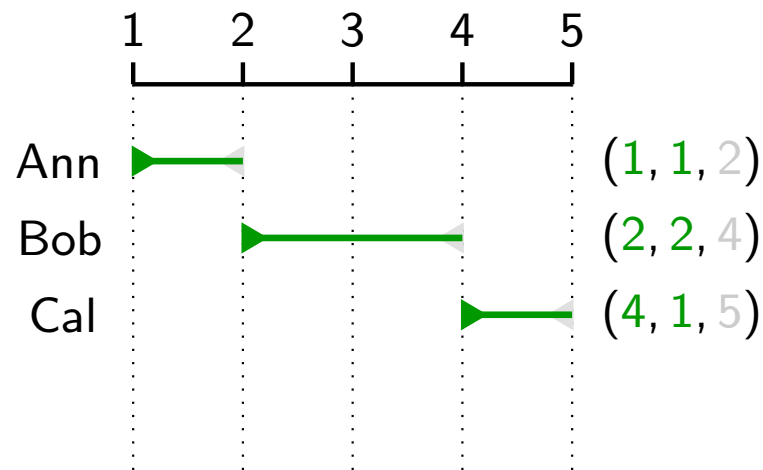
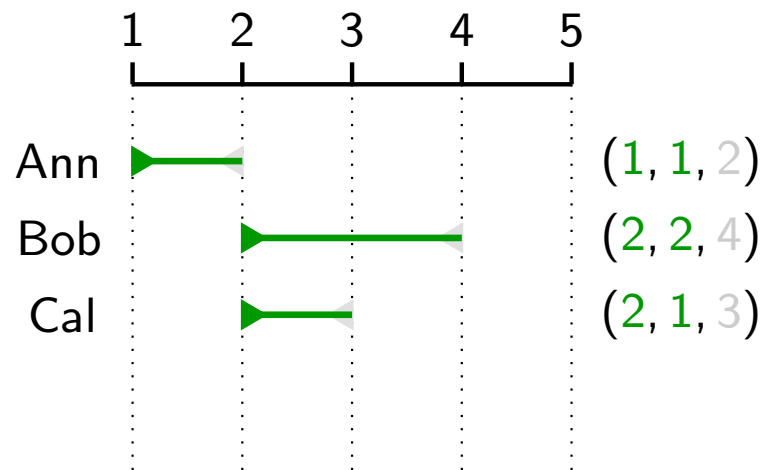
median-endpoint rule  $\rightarrow$  outcomes must have *different* widths

## Example



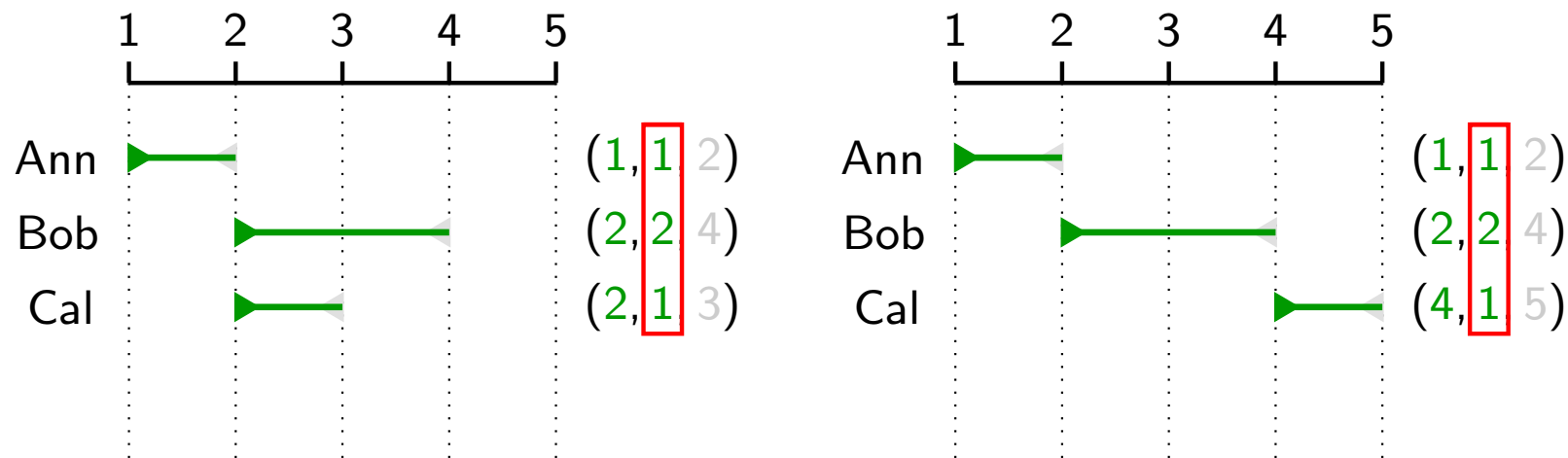
median-endpoint rule  $\rightarrow$  outcomes must have *different* widths

## Example



median-endpoint rule  $\rightarrow$  outcomes must have *different* widths

## Example



median-endpoint rule  $\rightarrow$  outcomes must have *different* widths  
 width-based representation  $\rightarrow$  outcomes must have *same* widths





## Talk Outline

So on our 5-point scale, the *median-endpoint rule* is *undefinable* when we represent intervals in terms of *left endpoints* and *widths*.

↪ *How general a problem is this?*

To find out, we shall see:

- Simple Model: *Interval Aggregation*
- New Concept: *Representation-Faithfulness*
- Results: *Impossibility Theorem* and *Characterisation Theorem*

## Interval Aggregation

Consider a *scale*  $S \subseteq \mathbb{R}$  of points (with a min- and a max-element).

Examples:  $S = \{-3, 0, 2, 4, 7, 10, 12\}$  or  $S = [0, 1]$

Several *agents*  $i \in N = \{1, \dots, n\}$  each report an *interval*  $I_i \in \mathcal{I}(S)$ .

We're interested in *interval aggregation rules*  $F : \mathcal{I}(S)^n \rightarrow \mathcal{I}(S)$ .

Examples: *medians of endpoints* or *convex hull of union*

## Representation-Faithfulness

We can talk about intervals by referring to their *components*, such as *left endpoint* ( $\ell$ ), *right endpoint* ( $r$ ), *midpoint* ( $m$ ), or *width* ( $w$ ).

A *representation formalism*  $\gamma = (\gamma_1, \dots, \gamma_q)$  is a list of such components  $\gamma_k : \mathcal{I}(S) \rightarrow D_k$  (for some domain  $D_k$ ) with  $[\gamma(I) = \gamma(I')] \Rightarrow [I = I']$ .

A rule  $F$  is *faithful* to  $(\gamma_1, \dots, \gamma_q)$  if we can define  $F$  via aggregators  $f_k : D_k^n \rightarrow D_k$  that each operate locally on just one component  $\gamma_k$ :

$$F \hat{=} (\gamma_1, \dots, \gamma_q) \circ (f_1, \dots, f_q)$$

Examples for natural rules:

- $F \hat{=} (\ell, r) \circ (\text{med}, \text{med})$
- $F \hat{=} (m, w) \circ (\text{avg}, \text{null})$
- $F \hat{=} (\ell, r) \circ (\text{min}, \text{max})$
- plurality (no natural representation!)

## Technical Results

*What rules can be defined by reference to left and right endpoints  $(\ell, r)$  and also by reference to left endpoints and widths  $(\ell, w)$ ?*

A  $(\gamma_1, \dots, \gamma_q)$ -*rule* is an aggregation rule that is *faithful* to  $(\gamma_1, \dots, \gamma_q)$  via *unanimous* local aggregators  $f_k$  (so: satisfying  $f_k(x, \dots, x) = x$ ).

**Impossibility Theorem:** *On **discrete scales**, every interval aggregation rule that is both an  $(\ell, r)$ - and an  $(\ell, w)$ -rule must a **dictatorship**.*

**Characterisation Theorem:** *On **continuous scales**, a continuous rule is both an  $(\ell, r)$ - and an  $(\ell, w)$ -rule iff it is a **weighted averaging rule**.*

## Proving the Characterisation Theorem

Let's understand this for scale  $S = [0, 1]$ ! (Generalisation not hard.)

**Characterisation Theorem:** *On scale  $S = [0, 1]$ , a continuous rule is both an  $(\ell, r)$ - and an  $(\ell, w)$ -rule iff it is a **weighted averaging rule**.*

So: Trying to understand, for any rule of  $F \hat{=} (\ell, r, w) \circ (f_\ell, f_r, f_w)$ , what options do we have for the local aggregators  $f_\ell, f_r, f_w$ ?

## First Insight: Just One Local Aggregator!

Inspecting specific scenarios unveils constraints:

- What if everyone submits intervals of width 0 (with  $\ell = r$ )?  
 $f_w$  is unanimous (so  $\ell = r$  also for outcome)  $\rightarrow f_\ell = f_r$
- What if everyone submits intervals that start at 0 (so  $w = r$ )?  
 $f_\ell$  is unanimous (so  $w = r$  also for outcome)  $\rightarrow f_w = f_r$

So we can focus on a single local aggregator:

$$f := f_\ell = f_r = f_w$$

## Interlude: Cauchy's Functional Equation

Which functions  $f : S \rightarrow S$  satisfy this for all  $x, y \in S$  with  $x + y \in S$ ?

$$f(x) + f(y) = f(x + y)$$

Cauchy answered this question for different choices of  $S$ . For  $S = [0, 1]$ , if  $f$  is continuous, then there must be some  $a \in [0, 1]$  such that:

$$f : x \mapsto a \cdot x$$

A.L. Cauchy. Cours d'Analyse de l'École Royale Polytechnique. I.<sup>re</sup> Partie: Analyse Algébrique. L'Imprimerie Royale, Paris, 1821.

## Second Insight: Apply Cauchy!

If our agents choose *left endpoints*  $x_1, \dots, x_n$  and *widths*  $y_1, \dots, y_n$ , then the *right endpoints* will be  $x_1 + y_1, \dots, x_n + y_n$ . Thus:

$$f(x_1, \dots, x_n) + f(y_1, \dots, y_n) = f(x_1 + y_1, \dots, x_n + y_n)$$

Now consider the case where all but agent  $i$  submit the interval  $[0, 0]$ :

$$f(\mathbf{0}_{-i}, x_i) + f(\mathbf{0}_{-i}, y_i) = f(\mathbf{0}_{-i}, x_i + y_i)$$

But this is an instance of Cauchy's functional equation! Thus:

$$f(\mathbf{0}_{-i}, z) = a_i \cdot z \quad (\text{for } a_i \in [0, 1])$$

But this fully determines  $f$ :

$$\begin{aligned} f(z_1, \dots, z_n) &= f(\mathbf{0}_{-1}, z_1) + \dots + f(\mathbf{0}_{-n}, z_n) \\ &= a_1 \cdot z_1 + \dots + a_n \cdot z_n \end{aligned}$$

Finally, due to *unanimity* of  $f$ , we must have  $a_1 + \dots + a_n = 1$ . ✓



## Back to the Impossibility Theorem

What is the connection between our two results?

- Characterisation: continuous scale  $\rightarrow$  only weighted averages work
- Impossibility: discrete scale  $\rightarrow$  only dictatorships work

Intuition: Dictatorships *are* weighted averages (dictator has weight 1), and they are the only ones that are well-defined on discrete scales.

However: Impossibility theorem *not* implied (due to restricted inputs).

Same proof technique works for the special case of “evenly-spaced” discrete scales. But for the full result, other techniques are needed.

## Message

*Representation matters*: the manner in which we *represent intervals* heavily constrains the *interval aggregation rules* we can design.

Our technical results concern endpoints-only vs. left endpoint + width:

- Characterisation: continuous scale → only weighted averages work
- Impossibility: discrete scale → only dictatorships work

The full paper is available online:

<http://bit.ly/interval-paper>

U. Endriss, A. Novaro, and Z. Terzopoulou. Representation Matters: Characterisation and Impossibility Results for Interval Aggregation. IJCAI-2022.