Vote Manipulation in the Presence of Multiple Sincere Ballots

Ulle Endriss
Institute for Logic, Language and Computation
University of Amsterdam

Talk Outline

- Background: the Gibbard-Satterthwaite Theorem . . . and why there is hope that it may, in some sense, not apply in *all* cases
- Background: Approval Voting (with multiple sincere ballots)
- Tie-Breaking and Preferences over Sets of Candidates
- Results: Manipulation in Approval Voting
- Conclusion

The Gibbard-Satterthwaite Theorem

Theorem 1 (Gibbard-Satterthwaite) *Every voting rule for three or more candidates must be either dictatorial or manipulable.*

Let C be a finite set of *candidates* and let $\mathcal{P}$ the set of all linear orders over C. A *voting rule* for n *voters* is a function $f : \mathcal{P}^n \to C$, selecting a *single winner* given the (reported) voter preferences.

A voting rule is *dictatorial* if the winner is always the top candidate of a particular voter (the dictator).

A voting rule is *manipulable* if there are situations where a (single) voter can force a preferred outcome by misreporting his preferences.

A. Gibbard. Manipulation of Voting Schemes: A General Result. *Econometrica*, 41(4):587–601, 1973.

M.A. Satterthwaite. Strategy-proofness and Arrow's Conditions. *Journal of Economic Theory*, 10:187–217, 1975.

The Gibbard-Satterthwaite Theorem

Theorem 1 (Gibbard-Satterthwaite) *Every voting rule for three or more candidates must be either dictatorial or manipulable.*

Despite its generality, the Gibbard-Satterthwaite Theorem may not apply in all cases (at least not immediately):

- The theorem presupposes that a ballot is a full preference ordering over all candidates. Plurality voting, for instance, does not satisfy this condition (although it's manipulable anyway).
- The theorem also presupposes that there is a *unique* way of casting a *sincere ballot* for any given preference ordering.

We will concentrate on the second "loophole". We can imagine several situations in which there may be more than one way of casting a sincere vote . . .

Approval Voting

In approval voting, a *ballot* is a subset of the set of candidates. These are the candidates the voter approves of. The candidate receiving the most approvals *wins* (we'll discuss tie-breaking later).

Approval voting has been used by several professional societies, such as the American Mathematical Society (AMS).

We assume each voter has a *preference ordering* $\preceq$ over candidates (which is antisymmetric, transitive and total).

A given voter's ballot is called *sincere* if all approved candidates are ranked above all disapproved candidates according to that voter's $\preceq$.

Example: If $A \succ B \succ C$, then $\{A\}$, $\{A, B\}$ and $\{A, B, C\}$ are all sincere ballots. The latter has the same effect as abstaining.

S.J. Brams and P.C. Fishburn. Approval Voting. *American Political Science Review*, 72(3):831–847, 1978.

Possible Tie-Breaking Rules

We call the candidates with the most approvals the *pre-winners*.

If there are two or more pre-winners, we have to use a suitable *tie-breaking rule* to choose a winner. Examples:

- The election chair may have the power to break ties.
- A designated voter may have the power to break ties.
- We may pick a winner from the set of pre-winners using a uniform probability distribution.
- We may pick a winner from the set of pre-winners using any other probability distribution.

We will try to avoid making too many assumptions as to which tie-breaking rule exactly is going to be used.

Axioms for Preferences over Sets of Candidates

Tie-breaking is outside the control of voters (in general). So when considering to manipulate, they have to do so in view of their preferences over sets of pre-winners.

Given a voter's preferences $\preceq$ over *individual candidates*, we assume that his preferences $\trianglelefteq$ over *sets of pre-winners* meet these axioms:

- $\trianglelefteq$ is reflexive and transitive.
- (DOM) $A \trianglelefteq B$ if $\#A = \#B$ and there exists a surjective mapping $f : A \rightarrow B$ such that $a \preceq f(a)$ for all $a \in A$.
- (ADD) $A \trianglelefteq B$ if $A \subset B$ and $a \preceq b$ for all $a \in A$ and all $b \in B \setminus A$.
- (REM) $A \trianglelefteq B$ if $B \subset A$ and $a \preceq b$ for all $a \in A \setminus B$ and all $b \in B$.

Note: It isn't impossible to conceive of tie-breaking rules that don't meet these axioms (e.g. when another agent chooses the winner).

An Example for Successful Manipulation

Suppose all but one voter have voted. This final voter wants to manipulate. His preferences are: $4 \succ 3 \succ 2 \succ 1$.

Suppose 3 and 1 each got 10 votes so far (*pivotal* candidates); 4 and 2 each got 9 (*subpivotal* candidates). The final voter can

- force outcome 431 by voting [4];
- force outcome 3 by voting [43], [432], [3] or [32];
- force outcome 31 by voting [4321], [431], [321] or [31];
- force outcome 4321 by voting [42];
- force outcome 1 by voting [421], [41], [21] or [1]; or
- force outcome 321 by voting [2].

Outcomes 431, 3 and 4321 are *undominated* according to our axioms. If (and only if) the final voter prefers 4321 amongst these, he has an incentive to submit the insincere ballot [42].

The Case of Optimistic Voters

We call a voter *optimistic* if his preferences over sets of pre-winners is induced only by his top candidate in each set:

$$A \trianglelefteq B \ \text{ iff } \ top(A) \preceq top(B) \quad [top(C) \in \{c^* \in C \mid \forall c \in C : c \preceq c^*\}]$$

Examples: "the election chair will break ties in my favour";
uniform tie-breaking + "extreme" utilities underlying $\preceq$.

Theorem 2 (Optimistic voters) *In approval voting, suppose that all but one voter have cast their ballot. Then, if the final voter is optimistic, he has no incentive to cast an insincere ballot.*

The Case of Three Candidates

Recall that the Gibbard-Satterthwaite hits once we move from two to three candidates. Our earlier example showed that approval voting is certainly manipulable in the case of four candidates . . .

Theorem 3 (Three candidates) *In approval voting with three candidates, suppose that all but one voter have cast their ballot. Then the final voter has no incentive to cast an insincere ballot.*

This is a special case of a result by Brams and Fishburn (1978).

S.J. Brams and P.C. Fishburn. Approval Voting. *American Political Science Review*, 72(3):831–847, 1978.

Proof of Theorem 3

Check all possible cases. For each candidate, distinguish whether she is pivotal (P), subpivotal (S) or insignificant (I). At least one has to be pivotal, so there are $3^3 - 2^3 = 19$ possible situations.

```
?- table(3).
-----------------------------------------------------
|     | [100] [110] [111] | [001] [010] [011] [101] |
-----------------------------------------------------
|   [... 9 obvious cases of the form P__ omitted]   |
| SPP |  321   2     21   |  1     2     21    1    |
| SPS |  32    2     2    |  21    2     2     321  |
| SPI |  32    2     2    |  2     2     2     32   |
| SSP |  31    321   1    |  1     21    1     1    |
| SIP |  31    31    1    |  1     1     1     1    |
| IPP |  21    2     21   |  1     2     21    1    |
| IPS |  2     2     2    |  21    2     2     21   |
| IPI |  2     2     2    |  2     2     2     2    |
| ISP |  1     21    1    |  1     21    1     1    |
| IIP |  1     1     1    |  1     1     1     1    |   ✓
-----------------------------------------------------
```

The Case of Four Candidates

We know that manipulation is possible with four candidates (see earlier example). But *how many* problematic situations are there?

Answer: Just one!

Theorem 4 (Four candidates) *In approval voting with four candidates, suppose that all but one voter have cast their ballot. Then the final voter has no incentive to cast an insincere ballot, unless he strictly prefers* 4321 *over both* 431 *and* 3*.*

The proof has been derived automatically using a *computer program* that checks all possible scenarios.

Question: Does this one exception to Theorem 4 matter in practice?

The Case of Expected-Utility Maximising Voters

If we are more specific about how ties are broken and how voters form preferences over sets of pre-winners we can obtain stronger results:

Theorem 5 (Expected-utility maximisers) *In approval voting with uniform tie-breaking, suppose that all but one voter have cast their ballot. Then, if the final voter is an expected-utility maximiser, he has no incentive to cast an insincere ballot.*

Conclusion

- Basic idea: The presence of *multiple sincere ballots* may allow us to circumvent the Gibbard-Satterthwaite Theorem in the sense that some sincere ballot may always be optimal.
- Results: For *approval voting*, it turns out that this is indeed the case for several interesting scenarios:
  - If all voters are *optimistic* (or pessimistic btw).
  - If there are at most *three* candidates.
  - If *uniform tie-breaking* is used and the voters are *expected-utility maximisers*.
  - More?