Representation Matters: Characterisation and Impossibility Results for Interval Aggregation

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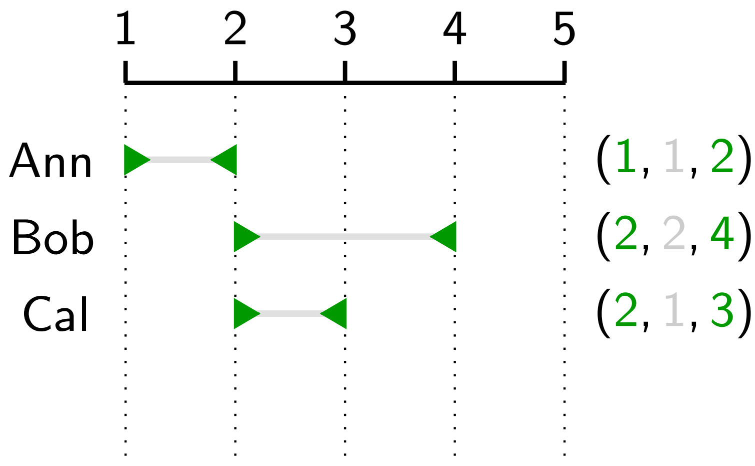
[joint work with Arianna Novaro and Zoi Terzopoulou]

Key Message

The *choice of components* we use to *represent intervals* heavily constrains the *range of aggregation rules* we can design.

Example



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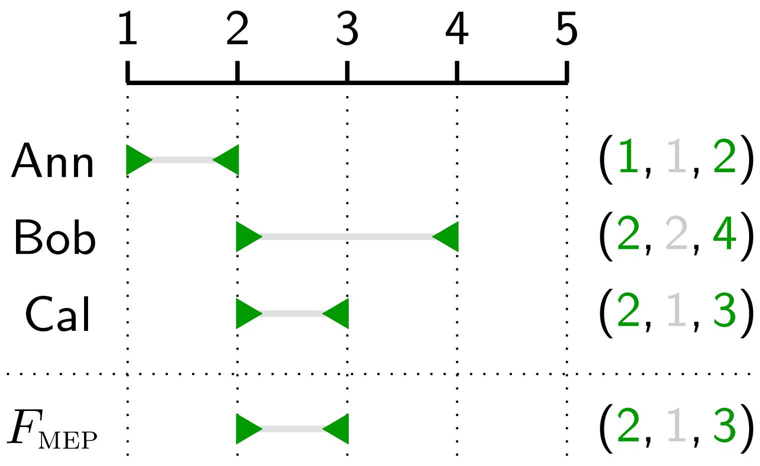

Let's use the median-endpoint rule to aggregate!

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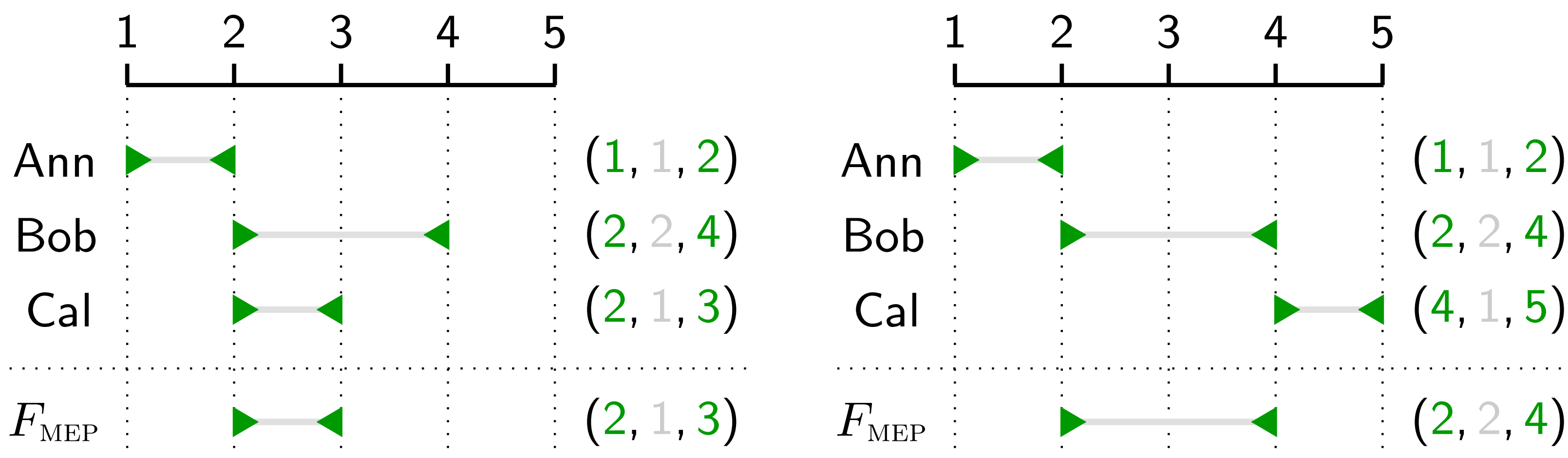

Let's use the median-endpoint rule to aggregate!

(now for both scenarios)

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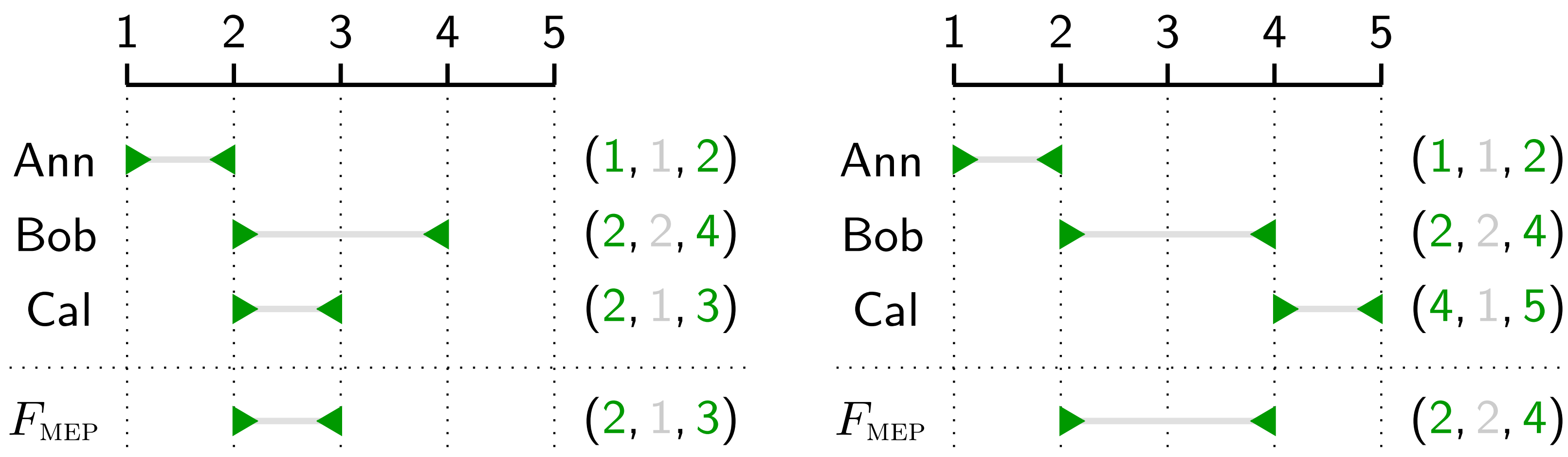

median-endpoint rule $\longrightarrow$ outcomes must have *different* widths

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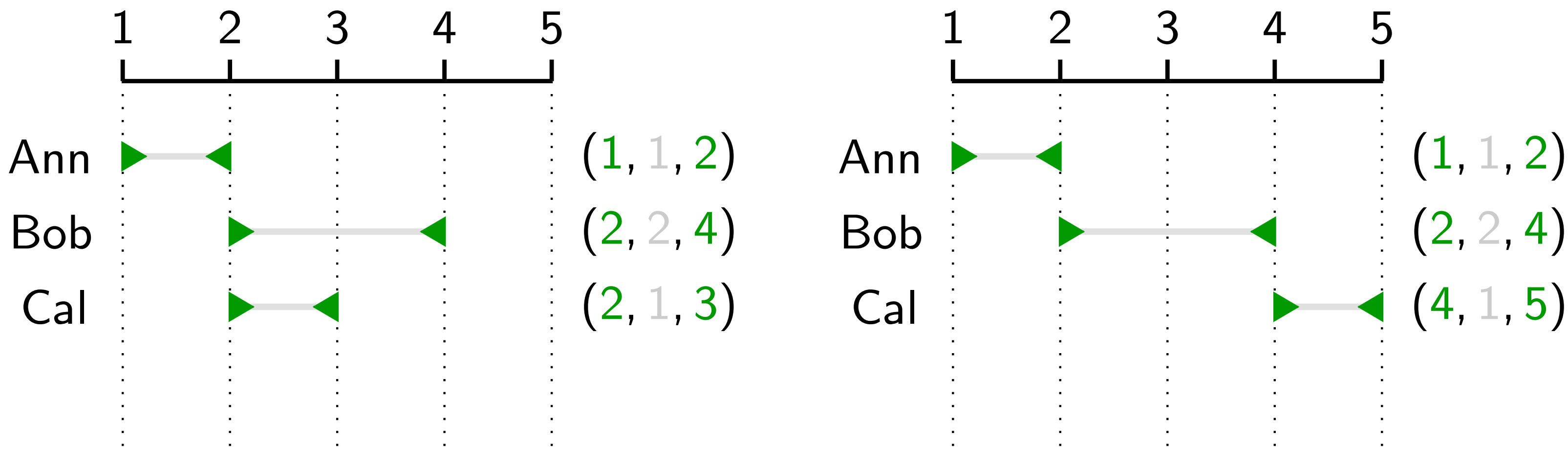

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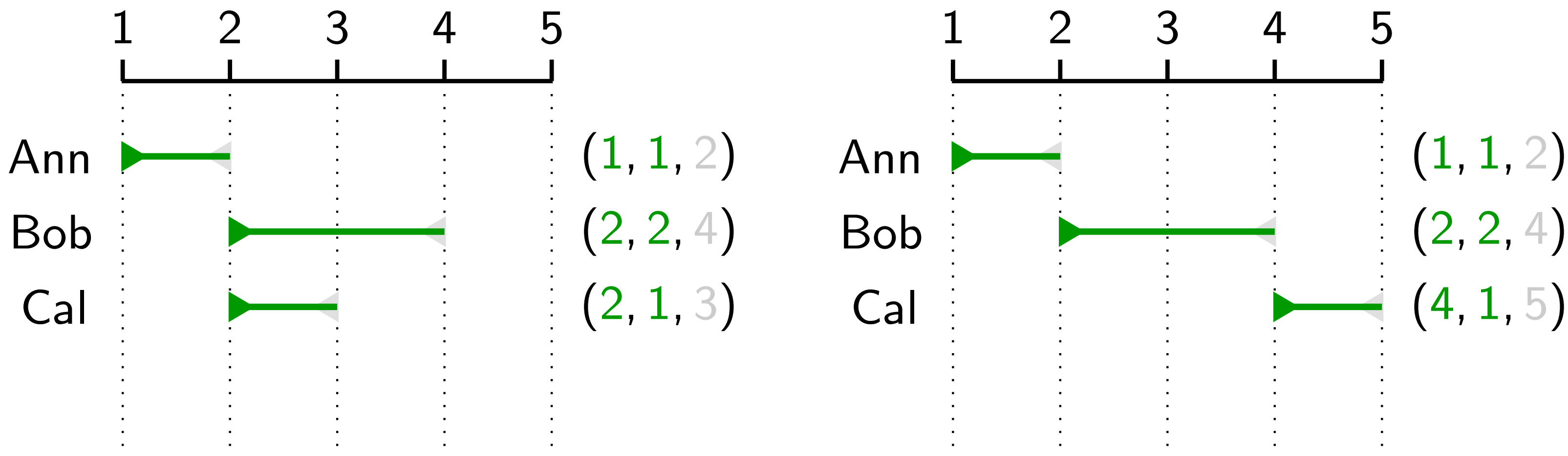

median-endpoint rule $\longrightarrow$ outcomes must have *different* widths

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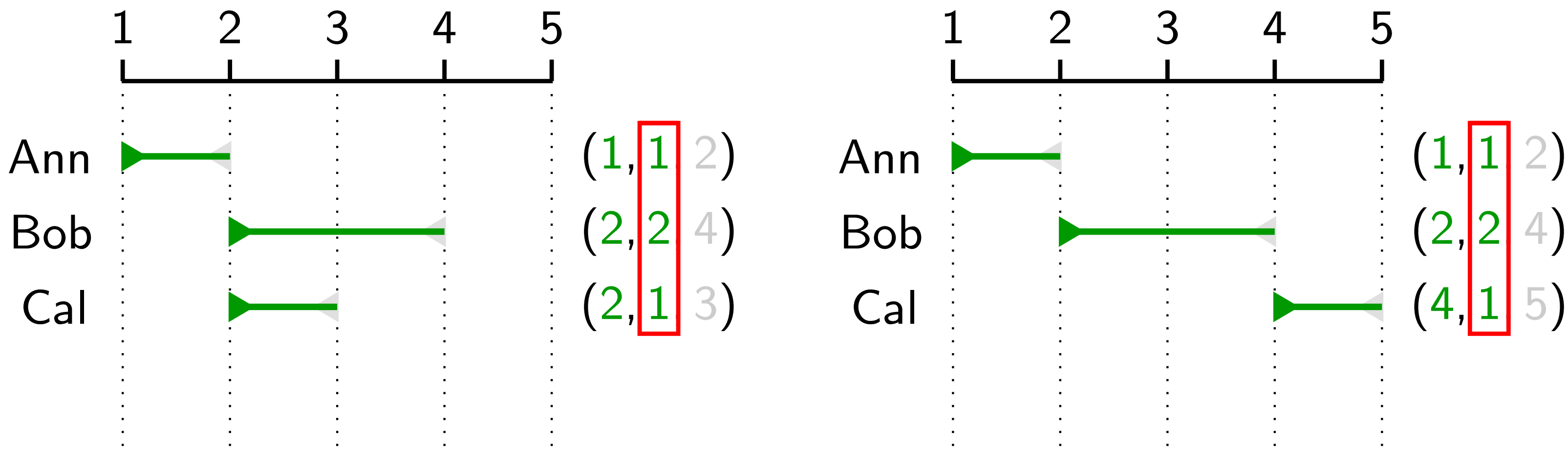

median-endpoint rule $\longrightarrow$ outcomes must have *different* widths

width-based representation $\longrightarrow$ outcomes must have *same* widths

Talk Outline

So on our 5-point scale, the *median-endpoint rule* is *undefinable* when we represent intervals in terms of *left endpoints* and *widths*.

$\hookrightarrow$ *How general a problem is this?*

To find out, we shall see:

- Simple Model: *Interval Aggregation*
- New Concept: *Representation-Faithfulness*
- Results: *Impossibility Theorem* and *Characterisation Theorem*

Interval Aggregation

Consider a *scale* $S \subseteq \mathbb{R}$ of points (with a min- and a max-element).

Examples: $S = \{-3, 0, 2, 4, 7, 10, 12\}$ or $S = [0, 1]$

Several *agents* $i \in N = \{1, \ldots, n\}$ each report an *interval* $I_i \in \mathcal{I}(S)$.
We're interested in *interval aggregation rules* $F : \mathcal{I}(S)^n \to \mathcal{I}(S)$.

Examples: *medians of endpoints* or *convex hull of union*

Representation-Faithfulness

We can talk about intervals by referring to their *components*, such as *left endpoint* (ℓ), *right endpoint* (r), *midpoint* (m), or *width* (w).

A *representation formalism* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_q)$ is a list of such components $\gamma_k : \mathcal{I}(S) \to D_k$ (for some domain D_k) with $[\boldsymbol{\gamma}(I) = \boldsymbol{\gamma}(I')] \Rightarrow [I = I']$.

A rule F is *faithful* to $(\gamma_1, \ldots, \gamma_q)$ if we can define F via aggregators $f_k : D_k^n \to D_k$ that each operate locally on just one component γ_k:

$$F \mathrel{\hat{=}} (\gamma_1, \ldots, \gamma_q) \circ (f_1, \ldots, f_q)$$

Examples for natural rules:

- $F \mathrel{\hat{=}} (\ell, r) \circ (\mathrm{med}, \mathrm{med})$
- $F \mathrel{\hat{=}} (\ell, r) \circ (\min, \max)$
- $F \mathrel{\hat{=}} (m, w) \circ (\mathrm{avg}, \mathrm{null})$
- plurality (no natural representation!)

Technical Results

What rules can be defined by reference to left and right endpoints (ℓ, r) and also by reference to left endpoints and widths (ℓ, w)?

A *$(\gamma_1, \ldots, \gamma_q)$-rule* is an aggregation rule that is *faithful* to $(\gamma_1, \ldots, \gamma_q)$ via *unanimous* local aggregators f_k (so: satisfying $f_k(x, \ldots, x) = x$).

Impossibility Theorem: *On discrete scales, every interval aggregation rule that is both an (ℓ, r)- and an (ℓ, w)-rule must a dictatorship.*

Characterisation Theorem: *On continuous scales, a continuous rule is both an (ℓ, r)- and an (ℓ, w)-rule iff it is a weighted averaging rule.*

Proving the Characterisation Theorem

Let's understand this for scale $S = [0,1]$! (Generalisation not hard.)

Characterisation Theorem: *On scale $S = [0,1]$, a continuous rule is both an (ℓ, r)- and an (ℓ, w)-rule iff it is a weighted averaging rule.*

So: Trying to understand, for any rule of $F \hat{=} (\ell, r, w) \circ (f_\ell, f_r, f_w)$, what options do we have for the local aggregators f_ℓ, f_r, f_w?

First Insight: Just One Local Aggregator!

Inspecting specific scenarios unveils constraints:

- What if everyone submits intervals of width 0 (with $\ell = r$)?
  f_w is unanimous (so $\ell = r$ also for outcome) $\rightarrow$ $f_\ell = f_r$
- What if everyone submits intervals that start at 0 (so $w = r$)?
  f_ℓ is unanimous (so $w = r$ also for outcome) $\rightarrow$ $f_w = f_r$

So we can focus on a single local aggregator:

$$f := f_\ell = f_r = f_w$$

Interlude: Cauchy's Functional Equation

Which functions $f : S \to S$ satisfy this for all $x, y \in S$ with $x + y \in S$?

$$f(x) + f(y) = f(x + y)$$

Cauchy answered this question for different choices of S. For $S = [0, 1]$, if f is continuous, then there must be some $a \in [0, 1]$ such that:

$$f : x \mapsto a \cdot x$$

A.L. Cauchy. Cours d'Analyse de l'École Royale Polytechnique. I.[re] Partie: Analyse Algébrique. L'Imprimerie Royale, Paris, 1821.

Second Insight: Apply Cauchy!

If our agents choose *left endpoints* $x_1, \ldots, x_n$ and *widths* $y_1, \ldots, y_n$, then the *right endpoints* will be $x_1 + y_1, \ldots, x_n + y_n$. Thus:

$$f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n) \;=\; f(x_1 + y_1, \ldots, x_n + y_n)$$

Now consider the case where all but agent i submit the interval $[0, 0]$:

$$f(\mathbf{0}_{-i}, x_i) + f(\mathbf{0}_{-i}, y_i) \;=\; f(\mathbf{0}_{-i}, x_i + y_i)$$

But this is an instance of Cauchy's functional equation! Thus:

$$f(\mathbf{0}_{-i}, z) \;=\; a_i \cdot z \quad (\text{for } a_i \in [0, 1])$$

But this fully determines f:

$$\begin{aligned} f(z_1, \ldots, z_n) &= f(\mathbf{0}_{-1}, z_1) + \cdots + f(\mathbf{0}_{-n}, z_n) \\ &= a_1 \cdot z_1 + \cdots + a_n \cdot z_n \end{aligned}$$

Finally, due to *unanimity* of f, we must have $a_1 + \cdots + a_n = 1$. ✓

Characterisation Theorem

We just saw that, on scale $S = [0, 1]$ a continuous interval aggregation rule is *both an (ℓ, r)- and an (ℓ, w)-rule* iff each of the three interval components is aggregated by the same function f:

$$f : (x_1, \ldots, x_n) \;\mapsto\; a_1 \cdot x_1 + \cdots + a_n \cdot x_n,$$

for some $a_1, \ldots, a_n \in [0, 1]$ *with* $a_1 + \cdots + a_n = 1$

So these conditions characterise the *weighted averaging rules*. ✓

Back to the Impossibility Theorem

What is the connection between our two results?

- Characterisation: continuous scale $\longrightarrow$ only weighted averages work
- Impossibility: discrete scale $\longrightarrow$ only dictatorships work

Intuition: Dictatorships *are* weighted averages (dictator has weight 1), and they are the only ones that are well-defined on discrete scales.

However: Impossibility theorem *not* implied (due to restricted inputs).

Same proof technique works for the special case of "evenly-spaced" discrete scales. But for the full result, other techniques are needed.

Message

Representation matters: the manner in which we *represent intervals* heavily constrains the *interval aggregation rules* we can design.

Our technical results concern endpoints-only *vs.* left endpoint + width:

- Characterisation: continuous scale $\longrightarrow$ only weighted averages work
- Impossibility: discrete scale $\longrightarrow$ only dictatorships work

U. Endriss, A. Novaro, and Z. Terzopoulou. Representation Matters: Characterisation and Impossibility Results for Interval Aggregation. IJCAI-2022.