Cake Cutting

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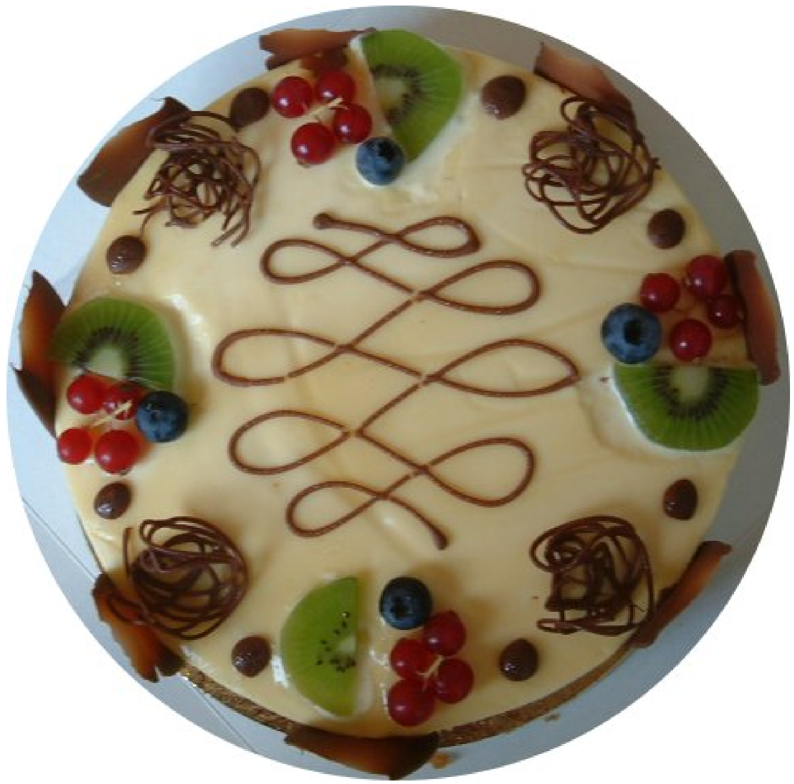
COMSOC and Cake Cutting

- **Social Choice Theory:** the study of mechanisms for collective decision making. Involves Political Science, Economics, Philosophy, and (some) Mathematics.
- **Computational Social Choice (COMSOC):** a broader view on social choice theory. Now also involves Computer Science, Artificial Intelligence, and (more) Logic.
- **Cake Cutting:** a nice example for a social choice problem.

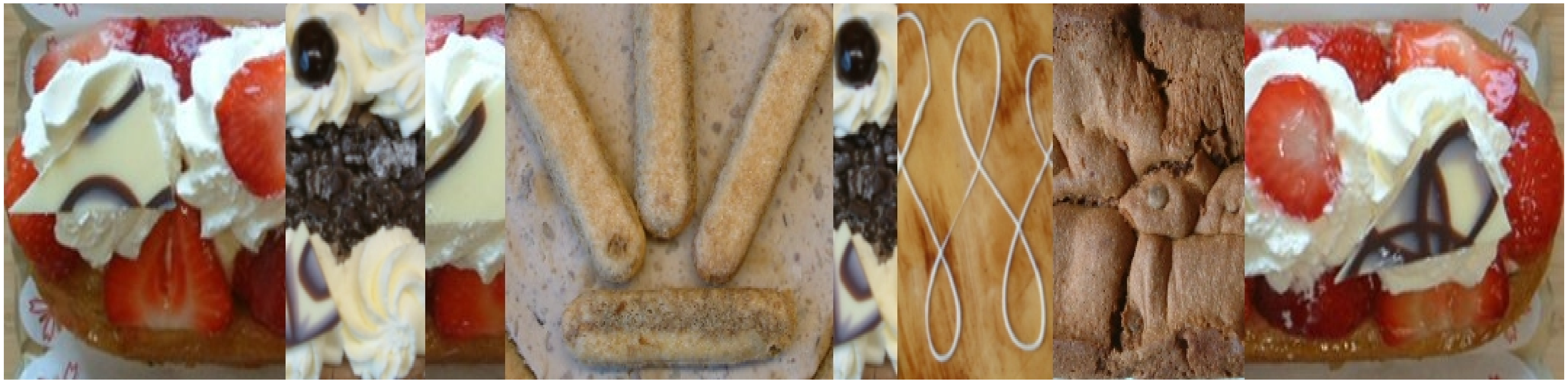
We practice what we preach.

What do I mean by "*cake*"?



Formal Model

There is a (heterogeneous, interval-shaped) *cake:*



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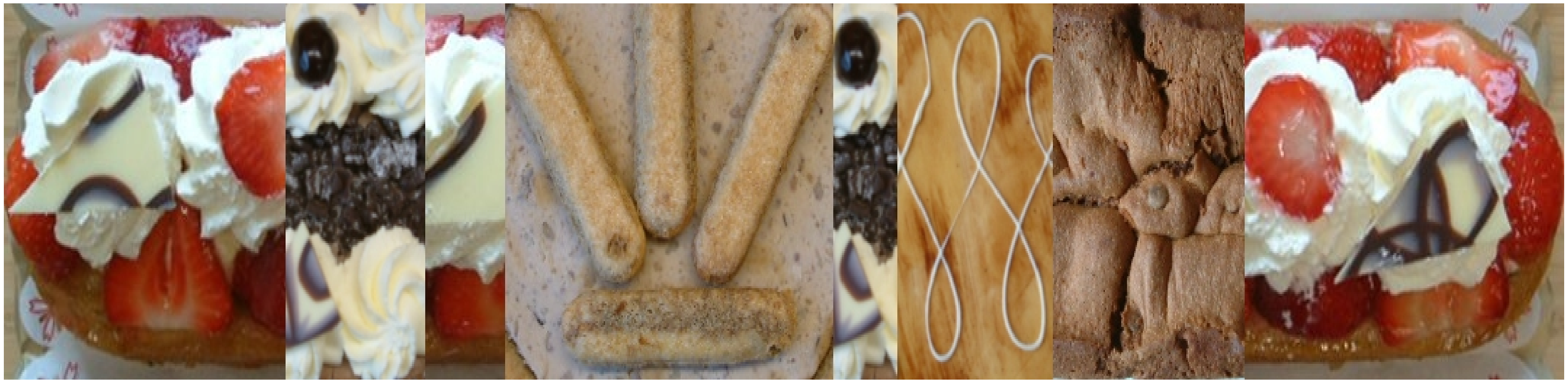

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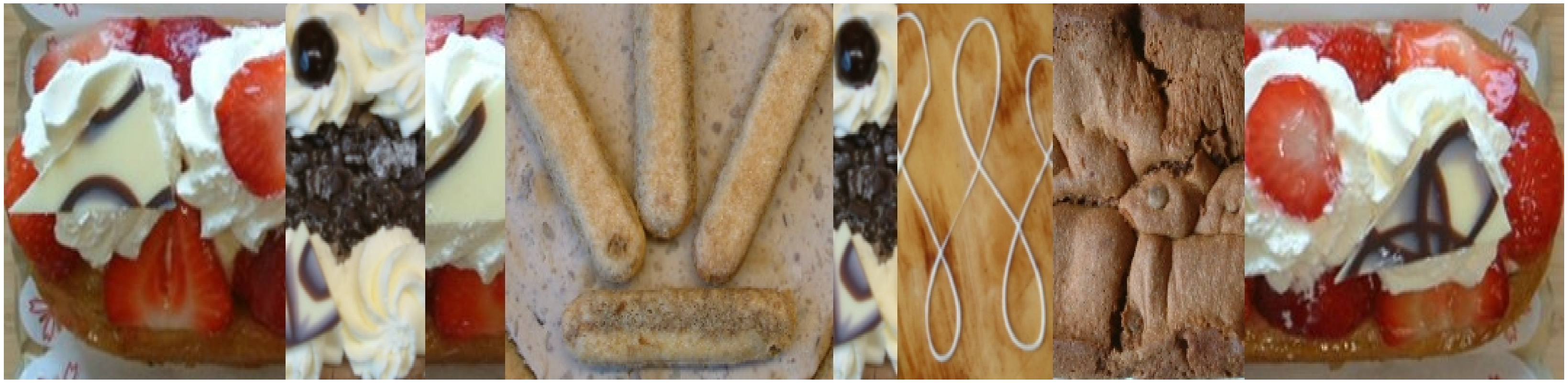

The cake needs to be divided amongst n *agents:*

Each agent i has *preferences* over what she will receive:

- valuation function v_i, mapping unions of subintervals to $[0, 1]$
- $v_i(\texttt{all}) = 1$ and $v_i(\texttt{nothing}) = 0$ / additive / continuous

The Cut-and-Choose Procedure

The classical approach for dividing a cake between *two agents:*

(1) One agent cuts the cake in two pieces (she assigns equal value to).

(2) The other one chooses one of the pieces (the one she prefers).

Cut-and-choose is *proportional:* each agent is guaranteed *at least* one half (general: $1/n$), according to her own valuation.

▶ *What about three or more agents?*

A Moving-Knife Procedure

Dubins and Spanier (1961) proposed a procedure for n *agents:*

(1) A referee moves a knife slowly across the cake, from left to right. Any agent may shout "stop" at any time. Whoever does so receives the piece to the left of the knife.

(2) When a piece has been cut off, we continue with the remaining $n-1$ agents, until just one agent is left (who takes the rest).

This procedure is also *proportional* (everyone is guaranteed $1/n$).

L.E. Dubins and E.H. Spanier. How to Cut a Cake Fairly. *American Mathematical Monthly*, 68(1):1–17, 1961.

Discretising the Moving-Knife Procedure

You cannot actually implement a moving-knife procedure!

But you can "*discretise*" it (this one, anyway):

(1) Ask each agent to suggest a cut.

(2) Realise the leftmost cut. Give the piece to whoever suggested it.

(3) When a piece has been cut off, we continue with the remaining $n-1$ agents, until just one agent is left (who takes the rest).

This procedure is also *proportional*.

Complexity analysis: $n + (n-1) + (n-2) + \cdots + 2 = O(n^2)$ operations

- *Can we do better?*

The Divide-and-Conquer Procedure

Even and Paz (1984) proposed another procedure for n *agents:*

(1) Ask each agent to cut the cake at ratio $\lfloor \frac{n}{2} \rfloor : \lceil \frac{n}{2} \rceil$.

(2) Associate the union of the leftmost $\lfloor \frac{n}{2} \rfloor$ pieces with the agents who made the leftmost $\lfloor \frac{n}{2} \rfloor$ cuts, and the rest with the others.

(3) Recursively apply the same procedure to each of the two groups, until only a single agent is left.

Also this procedure is *proportional*.

And it requires only $O(n \log n)$ operations.

But note that none of our procedures (for $n > 2$ agents) is *envy-free*.

S. Even and A. Paz. A Note on Cake Cutting. *Discrete Applied Mathematics*, 7(3):285–296, 1984.

Counting Envies

Can we guarantee a bound on the number of envy relations?

Theorem 1 (Brams et al., 2007) *Divide-and-conquer, after eliminating envy-cycles, produces at most* $\frac{(n-1)\cdot(n-2)}{2}$ *envies.*

Brams *et al*. use this, together with another result, to show that divide-and-conquer *minimises envy* wrt. a certain class of procedures.

We can *generalise* divide-and-conquer (and this result): take a tree with n leaves and associate each internal node with a local procedure.

Theorem 2 (E. & Pacuit, 2011) *A tree-based procedure with a minimum branching factor of* b *and locally envy-free local procedures, after eliminating envy-cycles, produces at most* $\frac{(n-1)\cdot(n-b)}{2}$ *envies.*

S.J. Brams, M.A. Jones, and C. Klamler. Divide-and-Conquer: A Proportional, Minimal-Envy Cake-Cutting Algorithm. *SIAM Review* (forthcoming).

U. Endriss and E. Pacuit. Tree-Based Cake-Cutting Procedures. Draft (2011).

Last Slide

- Cake Cutting: nice + simple framework, nontrivial problems
- We have seen some of the classical procedures for *proportional* division, and mentioned some of their properties.
- Many open problems: e.g., no known procedure for $n \geqslant 4$ that is *envy-free* and produces *contiguous* pieces (for $n = 3$ there is one, but it requires four simultaneously moving knives . . .).