Proportionality, Strategyproofness and Efficiency in Approval-Based Committee Voting

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Workshop on Social Choice Theory: "Strategyproofness and Beyond"

Preview

I will present an *impossibility theorem* regarding the integration of *proportionality*, *strategyproofness*, and *efficiency* requirements in the context of *multiwinner voting* with *approval ballots*.

I will be focusing on these aspects:

- suitable definitions of these *axioms* in this setting
- specific challenges raised by allowing for *irresolute* voting rules
- the use of computers (*SAT solvers*) to assist in the research process

Paper

This talk is based on published work done jointly with Boas Kluving, Adriaan de Vries, Pepijn Vrijbergen, and Arthur Boixel.

Our results were inspired by and generalise work by Dominik Peters, by moving from the case of *resolute* voting rules to *general* rules.

B. Kluiving et al. Analysing Irresolute Multiwinner Voting Rules with Approval Ballots via SAT Solving. Proceedings of ECAI-2020.

D. Peters. Proportionality and Strategyproofness in Multiwinner Elections. Proceedings of AAMAS-2018.

The Model: Approval-Based Committee Voting

Finite set $N = \{1, \ldots, n\}$ of voters who need to elect a *committee* X of size $k = |X|$ from a finite set C of *candidates* of size $m = |C|$.

Voters provide a *profile* $\boldsymbol{A} = (A_1, \ldots, A_n)$ of *approval ballots* $A_i \subseteq C$.

We are interested in (irresolute) *voting rules* F for this setting:

$$F : (2^C)^n \to 2^{[C]^k} \setminus \{\emptyset\}$$

Note that scenarios are characterised by three numbers: (n, m, k)

Examples of Voting Rules

The rule of *approval voting* elects the most popular candidates:

$$F_{\mathrm{AV}}(\boldsymbol{A}) \quad = \quad \operatorname*{argmax}_{X \in [C]^k} \sum_{i \in N} |A_i \cap X|$$

The rule of *proportional approval voting* tries to be more subtle:

$$F_{\mathrm{PAV}}(\boldsymbol{A}) \quad = \quad \operatorname*{argmax}_{X \in [C]^k} \sum_{i \in N} \sum_{\ell=1}^{|A_i \cap X|} \frac{1}{\ell}$$

The *sequential Phragmén rule* continuously releases a “budget” to all voters and makes groups “buy” candidates whenever they can afford it.

Axiom: Pareto Efficiency

Changing the committee elected should never make things better for some voters without also making things worse for some others.

But what does '*better*' mean in this context? Common choice:

> Voter $i \in N$ with truthfully held approval set $A_i \subseteq C$ cares only about the *number* of approved candidates elected:
>
> $$X \succcurlyeq_{A_i} X' \text{ if and only if } |A_i \cap X| \geqslant |A_i \cap X'|$$

<u>Discussion:</u> *Makes sense? Subset-based definition better?*

So let's call F *Pareto efficient* if $X \succcurlyeq_{A_i} X'$ for all voters $i \in N$ and $X \succ_{A_i} X'$ for at least one $i \in N$ together imply that $X' \notin F(\boldsymbol{A})$.

<u>Caveat:</u> Axiom not as weak as it may seem. Some popular ABC voting rules do not satisfy it (such as the sequential Phragmén rule).

Axiom: Strategyproofness

Voters should have no incentive to misrepresent their views: reporting an untruthful approval set should not result in a better outcome.

But an '*outcome*' here might be a *set of committees* (tied for winning). So we require voters to have preferences over those. Common choice:

> A *cautious* voter $i \in N$ with induced preference $\succcurlyeq_i$ is sure she weakly prefers outcome $\mathcal{X}$ to another outcome $\mathcal{X}'$ only if she would do so under any tie-breaking rule:

$$\mathcal{X} \succcurlyeq_{A_i}^{\text{CAU}} \mathcal{X}' \;\text{ if and only if }\; (\forall X \in \mathcal{X})\,(\forall X' \in \mathcal{X}')\, X \succcurlyeq_{A_i} X'$$

Remark: This is the well-known *Kelly extension*. Very weak (*good!*).

Discussion: *Other modelling options include optimism and pessimism.*

So let's call F is *strategyproof* if, for any voter $i \in N$ and any two profiles $\boldsymbol{A} =_{-i} \boldsymbol{A}'$, it is the case that $F(\boldsymbol{A}') \not\succ_{A_i}^{\text{CAU}} F(\boldsymbol{A})$.

Axiom: Proportionality

Sufficiently large groups of voters with sufficiently coherent views should get adequate representation in the committee elected.

But what do sufficiently '*coherent*' and '*large*' mean in this context? We can give an uncontroversial answer for this very special case:

> A group of voters who all want only c elected is coherent.
> It is sufficiently large if it represents one kth of the electorate.

So let's call F *proportional* if, for any candidate $c \in C$ and profile $\boldsymbol{A}$ with $|\{i \in N : A_i = \{c\}\}| \geqslant \frac{n}{k}$, we get $c \in X$ for all $X \in F(\boldsymbol{A})$.

Discussion: *Does this capture the intuition? It's certainly pretty weak.*

Remark: In the paper we use a weaker axiom (for *party-list profiles*).

Impossibility Theorem

We obtain the following unfortunate result:

> **Theorem:** *For $k \geqslant 3$, $m > k$, and $k \mid n$, there exists no voting rule that is Pareto efficient, strategyproof, and proportional.*

This is *essentially* a generalisation of a result due to Peters (2018) who considers *resolute* rules only (with $|F(\boldsymbol{A})| = 1$). Interesting differences:

- His efficiency axiom is genuinely weaker.
- His result holds even for subset-based preferences (ours does not).

D. Peters. Proportionality and Strategyproofness in Multiwinner Elections. Proceedings of AAMAS-2018.

Structure of the Proof

Write $\exists\textsc{Good}(n, m, k)$ if there exists a rule for dimensions (n, m, k) that is Pareto efficient, strategyproof, and proportional.

In the paper we prove three *inductive lemmas:*

- **Lemma 1:** $\exists\textsc{Good}(n, m+1, k) \Rightarrow \exists\textsc{Good}(n, m, k)$ *for* $m > k$
- **Lemma 2:** $\exists\textsc{Good}(n, k+1, k) \Rightarrow \exists\textsc{Good}(k, k+1, k)$ *for* $k \mid n$
- **Lemma 3:** $\exists\textsc{Good}(k+1, k+2, k+1) \Rightarrow \exists\textsc{Good}(k, k+1, k)$

So the impossibility theorem holds iff $\exists\textsc{Good}(3, 4, 3)$ is not the case.

Remark: Not all were easy to prove, but all are very *unsurprising*.

Proof of the Base Case

Lemma: *For $n = 3$, $m = 4$, and $k = 3$, there exists no voting rule that is Pareto efficient, strategyproof, and proportional.*

Proof: Consider the following three profiles.

$$
\begin{array}{lcll}
\boldsymbol{A} & = & \{\, \{c,d\}, \{a,c,d\}, \{b,c,d\} \,\} & \\
\boldsymbol{A}' & = & \{\, \{c,d\}, \{a\}, \qquad \{b,c,d\} \,\} & a \text{ wins by proportionality} \\
\boldsymbol{A}'' & = & \{\, \{c,d\}, \{a,c,d\}, \{b\} \qquad \,\} & b \text{ wins by proportionality}
\end{array}
$$

By Pareto efficiency, c and d win (are elected) in all three profiles.

Consider voter 2 in profile $\boldsymbol{A}$. Due to strategyproofness, we can't allow $\{b, c, d\} \in F(\boldsymbol{A})$, as she would move to $\boldsymbol{A}'$ with $F(\boldsymbol{A}') = \{\{a, c, d\}\}$.

Consider voter 3 in profile $\boldsymbol{A}$. Due to strategyproofness, we can't allow $\{a, c, d\} \in F(\boldsymbol{A})$, as she would move to $\boldsymbol{A}''$ with $F(\boldsymbol{A}'') = \{\{b, c, d\}\}$.

Thus: $F(\boldsymbol{A}) = \emptyset$. *Contradiction!* ✓

Methodology: Automated Theorem Proving

While the inductive lemmas were difficult but arguably *unsurprinsing*, the base case lemma was easy (once you see it!) but pretty *surprising*.

We tried *many* variations of axioms and combinations thereof ...

We used the *SAT solving approach* to SCT to structure our search:

- express all desiderata for F for $(3, 4, 3)$ in propositional logic
- use a SAT solver to show that their conjunction is unsatisfiable
- use an MUS extractor to understand why that is so

For example, *strategyproofness* can be expressed like this:

$$\varphi_{\mathrm{SP}} \quad = \quad \bigwedge_{i \in N} \; \bigwedge_{\boldsymbol{A} =_{-i} \boldsymbol{A}'} \; \bigwedge_{\mathcal{X}' \succ^{\mathrm{CAU}}_{A_i} \mathcal{X}} \mathrm{DIFF}(\boldsymbol{A}, \mathcal{X}) \;\vee\; \mathrm{DIFF}(\boldsymbol{A}', \mathcal{X}')$$

$$\text{where } \mathrm{DIFF}(\boldsymbol{A}, \mathcal{X}) = \bigvee_{X \in \mathcal{X}} \neg p_{\boldsymbol{A},X} \;\vee \bigvee_{X \in [C]^k \setminus \mathcal{X}} p_{\boldsymbol{A},X}$$

That's $10,838,016$ clauses for $(3, 4, 3)$. But computers can do this.

Advantages of the SAT Solving Approach

- Easy to experiment with variants of the axioms to get proofs for new base cases and thus strong conjectures for new theorems.
- Easy to check for counterexamples for impossibilities for weaker axioms (though *not* easy to interpret the counterexamples found).

Last Slide

I discussed three appealing properties of approval-based committee voting rules and explained why they cannot be satisfied together:

efficiency — *strategyproofness* — *proportionality*

Questions: *Should one insist on (this variant of) efficiency? If not, can we get a possibility result?* [We have an artificial rule for $(3, 4, 3)$.]

Methodology: A crucial step in the proof was *found* with the help of a *SAT solver* (but can easily be *verified* without the help of computers!).

Credo: I see a lot of potential in the SAT approach, for all of SCT.

Paper and code are available here:

`http://bit.ly/abc-voting-analysis`