Complexity of Judgment Aggregation

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The Paradox of Judgment Aggregation

Story: three judges have to decide whether the defendant is guilty ...

	p	$p \to q$	q
Judge 1:	Yes	Yes	Yes
Judge 2:	No	Yes	No
Judge 3:	Yes	No	No
Majority:	Yes	Yes	No

Paradox: each *individual* judgment set is *consistent*, but the *collective* judgment arrived at using the *majority rule* is not

L.A. Kornhauser and L.G. Sager. The One and the Many: Adjudication in Collegial Courts. *California Law Review*, 81(1):1–59, 1993.

C. List and C. Puppe. Judgment Aggregation: A Survey. *Handbook of Rational and Social Choice*. Oxford University Press, 2009.

Talk Outline

- Introduction to Judgment Aggregation
- Decision Problems in JA and their Complexity:
  - Winner Determination
  - Strategic Manipulation
  - Safety of the Agenda
- More on Computational Social Choice

Formal Framework

An *agenda* Φ is a finite nonempty set of propositional formulas (w/o double negation) closed under complementation: $\varphi \in \Phi \ \Rightarrow \ {\sim}\varphi \in \Phi$.

A *judgment set* J on an agenda Φ is a subset of Φ. We call J:

- *complete* if $\varphi \in J$ or ${\sim}\varphi \in J$ for all $\varphi \in \Phi$
- *complement-free* if $\varphi \notin J$ or ${\sim}\varphi \notin J$ for all $\varphi \in \Phi$
- *consistent* if there exists an assignment satisfying all $\varphi \in J$

Let $\mathcal{J}(\Phi)$ be the set of all complete and consistent subsets of Φ.

Now a finite set of *individuals* $\mathcal{N} = \{1, \ldots, n\}$ express judgments on the formulas in Φ, giving rise to a *profile* $\mathbf{J} = (J_1, \ldots, J_n)$.

An *aggregation procedure* for agenda Φ and a set of n individuals is a function mapping a profile of complete and consistent individual judgment sets to a single collective judgment set: $F : \mathcal{J}(\Phi)^n \rightarrow 2^\Phi$.

Premise-Based Procedure

Suppose we *can* divide the agenda into *premises* and *conclusions:*

$$\Phi \quad = \quad \Phi_p \uplus \Phi_c$$

The *premise-based procedure* PBP for Φ_p and Φ_c is this function:

$$\begin{aligned} \mathrm{PBP}(\mathbf{J}) \quad &= \quad \Delta \cup \{\varphi \in \Phi_c \mid \Delta \models \varphi\}, \\ &\quad\quad \text{where } \Delta = \{\varphi \in \Phi_p \mid \#\{i \mid \varphi \in J_i\} > \frac{n}{2}\} \end{aligned}$$

If we assume (as we shall) that

- the set of premises is the set of literals in the agenda,
- the agenda Φ is is closed under propositional letters, and
- the number n of individuals is odd,

then $\mathrm{PBP}(\mathbf{J})$ will always be *consistent* and *complete*.

Winner Determination

The winner determination problem for a judgment aggregation procedure F is defined as follows:

$\textsc{WinDet}(F)$
Instance: Agenda Φ, profile $\mathbf{J} \in \mathcal{J}(\Phi)^n$, formula $\varphi \in \Phi$.
Question: Is φ an element of $F(\mathbf{J})$?

This is easy under the (simplified) premise-based procedure:

Proposition 1 $\textsc{WinDet}(\mathrm{PBP})$ *is in P.*

Proof: some counting + model checking. ✓

Example: Strategic Manipulation

Suppose we use the (simplified) premise-based procedure:

	p	q	$p \vee q$
Agent 1:	No	No	No
Agent 2:	Yes	No	Yes
Agent 3:	No	Yes	Yes

If agent 3 only cares about the conclusion, then she has an incentive to *manipulate* and pretend she accepts p.

Strategic Manipulation

Let us fix a notion of *preference* over outcomes:

- The *Hamming distance* $H(J, J')$ between judgment sets J and J' is the number of positive agenda formulas on which they differ.
- We say that individual i *prefers* J to J' if $H(J_i, J) < H(J_i, J')$.

Now we can define the manipulability problem:

$\textsc{Manipulability}(F)$
Instance: Agenda Φ, $J_i \in \mathcal{J}(\Phi)$, partial profile $\mathbf{J}_{-i} \in \mathcal{J}(\Phi)^{n-1}$.
Question: Is there a $J_i' \in \mathcal{J}(\Phi)$ s.t. i prefers $F(J_i', \mathbf{J}_{-i})$ to $F(J_i, \mathbf{J}_{-i})$?

Good news:

Theorem 2 $\textsc{Manipulability}(\text{PBP})$ *is NP-complete.*

Proof by reduction from $\textsc{Sat}$.

Distance-Based Procedure

A procedure that is more widely applicable than the premise-based procedure and that is intuitively appealing is *distance-based merging:*

$$\mathrm{DBP}(\mathbf{J}) \quad = \quad \operatorname*{arg\,min}_{J \in \mathcal{J}(\Phi)} \sum_{i=1}^{n} H(J, J_i)$$

Remark: The DBP may return a set of tied winners.

Regarding complexity, we have:

Theorem 3 *Winner determination for the DBP is NP-complete.*

Proof of hardness by reduction from a result in computational social choice (KemenyScore). Membership via an integer program.

Conjecture 4 *Manipulability for the DBP is Σ_2^p-complete.*

Membership is clear. Hardness is open.

The Axiomatic Method

What makes for a "good" aggregation procedure? The standard approach in social choice theory is to formulate "*axioms*", e.g.:

Unanimity (U): If $\varphi \in J_i$ for all i, then $\varphi \in F(\mathbf{J})$.

Anonymity (A): For any profile $\mathbf{J}$ and any permutation $\sigma : \mathcal{N} \rightarrow \mathcal{N}$ we have $F(J_1, \ldots, J_n) = F(J_{\sigma(1)}, \ldots, J_{\sigma(n)})$.

Neutrality (N): For any φ, ψ in the agenda Φ and profile $\mathbf{J} \in \mathcal{J}(\Phi)$, if for all i we have $\varphi \in J_i \Leftrightarrow \psi \in J_i$, then $\varphi \in F(\mathbf{J}) \Leftrightarrow \psi \in F(\mathbf{J})$.

Independence (I): For any φ in the agenda Φ and profiles $\mathbf{J}$ and $\mathbf{J}'$ in $\mathcal{J}(\Phi)$, if $\varphi \in J_i \Leftrightarrow \varphi \in J'_i$ for all i, then $\varphi \in F(\mathbf{J}) \Leftrightarrow \varphi \in F(\mathbf{J}')$.

Impossibility Theorem

We have seen that the majority rule is not consistent.

Is there a reasonable procedure that is?

Theorem 5 (List and Pettit, 2002) *If the agenda contains at least* p*,* q *and* $p \wedge q$*, then no aggregation procedure producing consistent and complete judgment sets satisfies all of* (A), (N) *and* (I)*.*

C. List and P. Pettit. Aggregating Sets of Judgments: An Impossibility Result. *Economics and Philosophy*, 18(1):89–110, 2002.

More Axioms

Two monotonicity axioms, one for independent rules (inter-profile) and one for neutral rules (intra-profile):

I-Monotonicity (M^I): For any φ in the agenda Φ and profiles $\mathbf{J} = (J_1, \ldots, J_i, \ldots, J_n)$ and $\mathbf{J}' = (J_1, \ldots, J_i', \ldots, J_n)$ in $\mathcal{J}(\Phi)$, if $\varphi \notin J_i$ and $\varphi \in J_i'$, then $\varphi \in F(\mathbf{J}) \Rightarrow \varphi \in F(\mathbf{J}')$.

N-Monotonicity (M^N): For any φ, ψ in the agenda Φ and profile $\mathbf{J}$ in $\mathcal{J}(\Phi)$, if $\varphi \in J_i \Rightarrow \psi \in J_i$ for all i and $\varphi \notin J_k$ and $\psi \in J_k$ for some k, then $\varphi \in F(\mathbf{J}) \Rightarrow \psi \in F(\mathbf{J})$.

Ideally, we'd like consistent outcomes, but instead we just demand:

Weak Rationality (WR): $F(\mathbf{J})$ is complete and complement-free for *all* profiles $\mathbf{J}$ [and $F(\mathbf{J})$ includes no contradictions for *some* $\mathbf{J}$]

Remark: the last condition ("non-nullity") is a minor technicality (always satisfied if Φ includes no tautologies) — please ignore

Safety of the Agenda (SoA)

Given an agenda Φ and a list of axioms AX, let $\mathcal{F}_\Phi[\text{AX}]$ be the set of procedures $F : \mathcal{J}(\Phi)^n \to 2^\Phi$ that satisfy all axioms in AX.

We call an agenda Φ is *safe* wrt. a class of procedures $\mathcal{F}_\Phi[\text{AX}]$, if $F(\mathbf{J})$ is consistent for every $F \in \mathcal{F}_\Phi[\text{AX}]$ and every $\mathbf{J} \in \mathcal{J}(\Phi)$.

Goal: We want to be able to check the safety of a given agenda for a given class of procedures (characterised in terms of a set of axioms).

We approach this by proving *characterisation results:*

all $F \in \mathcal{F}_\Phi[\text{AX}]$ are consistent $\Leftrightarrow$ Φ has such-and-such property

This is similar to *possibility results* proven in the JA literature:

some $F \in \mathcal{F}_\Phi[\text{AX}]$ is consistent $\Leftrightarrow$ Φ has such-and-such property

Majority Rule

It is known (Nehring and Puppe, 2007) that the *majority rule* is consistent on agendas that satisfy the *median property*.

> Φ *satisfies the median property (MP), if every inconsistent subset of* Φ *has itself an inconsistent subset of size* $\leqslant 2$.

It is also known (folk theorem?) that

$$\mathcal{F}_\Phi[\mathsf{WR,A,N,I,M^I}] = \mathcal{F}_\Phi[\mathsf{WR,A,N,M^N}] = \{\text{majority rule}\}$$

We thus get our first characterisation theorem for free:

Theorem 6 Φ *is safe for* $\mathcal{F}_\Phi[\mathsf{WR,A,N,I,M^I}]$ *(and thus also for* $\mathcal{F}_\Phi[\mathsf{WR,A,N,M^N}]$*) iff it satisfies the MP.*

K. Nehring and C. Puppe. The Structure of Strategy-proof Social Choice. *Journal of Economic Theory*, 135(1):269–305, 2007.

Agenda Properties

Call a set of formulas *nontrivially inconsistent* if it is inconsistent but does not contain an inconsistent formula. An agenda Φ satisfies

- the *median property* (MP), if every nontrivially inconsistent subset of Φ has itself an inconsistent subset of size 2;
- the *simplified MP* (SMP), if every nontrivially inconsistent subset of Φ has itself an inconsistent subset $\{\varphi, \psi\}$ with $\models \varphi \leftrightarrow \neg\psi$;
- the *syntactic SMP* (SSMP), if every nontrivially inconsistent subset of Φ has itself an inconsistent subset $\{\varphi, \neg\varphi\}$.

$$\text{SSMP} \Rightarrow \text{SMP} \Rightarrow \text{MP}$$

Characterisation Theorems

We have looked for characterisation theorems for sets of axioms that are a little weaker than those defining the majority rule.

Theorem 7 *Φ is safe for $\mathcal{F}_\Phi$[WR,A,N,I] iff it satisfies the SMP.*

Theorem 8 *Φ is safe for $\mathcal{F}_\Phi$[WR,A,N] iff it satisfies the SMP and does not contain a contradictory formula.*

Theorem 9 *Φ is safe for $\mathcal{F}_\Phi$[WR,A,I] iff it satisfies the SSMP.*

Complexity Results

For a given agenda, how hard is it to check safety?

Theorem 10 *Checking the safety of the agenda is Π_2^p-complete for any of the classes of aggregation procedures considered.*

Approach:

- the typical Π_2^p-complete problem is $\mathrm{S{\small AT}}$ for QBFs of the form

$$\forall x_1 \cdots x_r \exists y_1 \cdots y_s . \varphi(x_1, \ldots, x_r, y_1, \ldots, y_s)$$

- reduce that problem to the problem of checking the SSMP, to establish Π_2^p-*hardness* of the latter (similarly for SMP and MP)
- prove that checking the SSMP, SMP, MP are all *in* Π_2^p
- apply the characterisation theorems

Last Slide

- We have seen several results in *judgment aggregation:*
  - *Manipulation* tends to be harder than *winner determination* (good)
  - *SoA* requires simplistic agendas and is hard to check (bad)
- This is an example for work in *Computational Social Choice*, combining ideas from economics (particularly social choice theory) and CS.
  - SCT: preference aggregation, voting, fair division, matching, ...
  - CS: algorithms, complexity, logic, knowledge representation, ...

  For more information, see the COMSOC website:

  `http://www.illc.uva.nl/COMSOC/`

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