Graph Aggregation

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Talk Outline

- Graph aggregation
- Collective rationality wrt. some property of graphs
- Possibility and impossibility results

Graph Aggregation

Finite set of *vertices* V ($|V| \geqslant 3$). Set $\mathcal{G}$ of *directed graphs* $G = \langle V, E \rangle$.
Finite set of *agents* $\mathcal{N} = \{1, \ldots, n\}$, each providing one such graph.

How should we aggregate this profile into a single collective graph?

- We will study *aggregators* $F : \mathcal{G}^{\mathcal{N}} \to \mathcal{G}$.

We might want to impose certain *axioms* on F, such as:

- *anonymity*: $F(G_1, \ldots, G_n) = F(G_{\pi(1)}, \ldots, G_{\pi(n)})$
- *unanimity*: $E \supseteq E_1 \cap \cdots \cap E_n$
- *groundedness*: $E \subseteq E_1 \cup \cdots \cup E_n$
- *neutrality*: $N_e^{\boldsymbol{G}} = N_{e'}^{\boldsymbol{G}}$ implies $e \in F(\boldsymbol{G}) \Leftrightarrow e' \in F(\boldsymbol{G})$
- *independence*: $N_e^{\boldsymbol{G}} = N_e^{\boldsymbol{G'}}$ implies $e \in F(\boldsymbol{G}) \Leftrightarrow e \in F(\boldsymbol{G'})$

Notation: $N_e^{\boldsymbol{G}}$ is the set of agents accepting edge e in profile $\boldsymbol{G}$.

Collective Rationality

An aggregator F is *collectively rational* wrt. a property P if every individual graph in profile $\boldsymbol{G}$ satisfying P implies the same for $F(\boldsymbol{G})$.

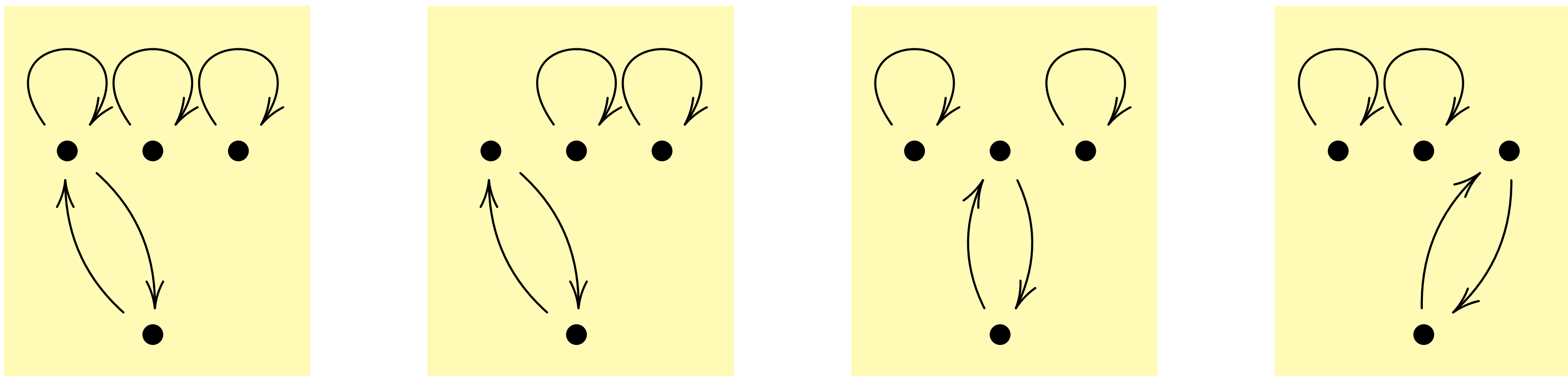
What properties to consider?

We'll take our inspiration from *modal logic* (graphs $=$ *Kripke frames*):

- Consider *frame properties* typically discussed in correspondence theory: reflexivity, symmetry, transitivity, seriality, ...
- Consider the property of satisfying a given *modal formula* in a given state (precise definition later).

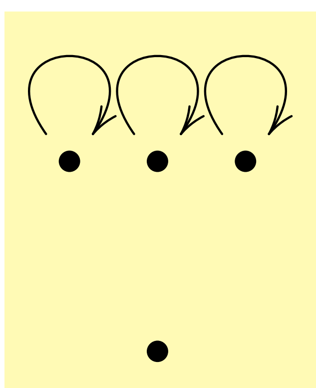
Example

Four agents each provide a graph on the same set of four vertices:



1 2 3 4

If we aggregate using the *strict majority rule* (SMR), we obtain:



Observations:

- SMR not collectively rational wrt. *seriality*.
- But *symmetry* is preserved.
- So is *reflexivity* (easy: individuals violate it).

A Simple Possibility Result

The fact that the example worked for reflexivity is no coincidence:

Proposition 1 *Any unanimous aggregator is CR wrt. reflexivity.*

Proof: If every individual graph includes edge (x, x), then unanimity ensures the same for the collective graph. ✓

A Negative Result for Quota Rules

A *uniform quota rule* (with quota q) accepts an edge if $\geqslant q$ agents do. The rules with $q = 0$ and $q = n{+}1$ are called *trivial*.

Recall: connectedness means $\forall xyz.[(xEy \wedge xEz) \rightarrow (yEz \vee zEy)]$

Proposition 2 *No nontrivial uniform QR is CR wrt. connectedness.*

Proof: Take any quota q with $0 < q < n$ (proof for $q = n$ is similar).

Consider this scenario:

- a group of q agents accept edge (x, y)
- a *different* group of q agents accept edge (x, z)
- everyone in the intersection accepts (y, z) [connectedness ✓]
- nobody accepts (z, y)

Then both (x, y) and (x, z) but neither (y, z) or (z, y) belong to the collective graph, as the group accepting (y, z) doesn't make the quota. This violates connectedness. ✓

Winning Coalitions

If an aggregator F is *independent*, then for every edge e there exists a set of *winning coalitions* $\mathcal{W}_e \subseteq 2^{\mathcal{N}}$ such that $e \in F(\boldsymbol{G}) \Leftrightarrow N_e^{\boldsymbol{G}} \in \mathcal{W}_e$.

Furthermore:

- If F is *unanimous*, then $\mathcal{N} \in \mathcal{W}_e$ for any edge e.
- If F is *grounded*, then $\emptyset \notin \mathcal{W}_e$ for any edge e.
- If F is *neutral*, then there is a $\mathcal{W}$ with $\mathcal{W} = \mathcal{W}_e$ for any edge e.

A Nice Lemma

Lemma 3 *Any unanimous and independent aggregator that is CR wrt. transitivity must be neutral [wrt. nonreflexive edges].*

Proof sketch: [Independence = F can be described in terms of a set of winning coalitions for each edge e. Neutrality = they are all the same.]

Proof for the typical case:

Suppose *C is a winning coalition for (x, y)*.

Suppose (only) agents in C accept (x, y) and (x', y') and everyone accepts (x', x) and (y, y'). [Individual transitivity ok.]

As C is a winning coalition for (x, y), it is collectively accepted. By unanimity, (x', x) and (y, y') are as well.

Collective transitivity now forces collective acceptance of (x', y'). Hence, *C is a winning coalition for (x', y')* as well. ✓

Also works for *Euclidean property* ($\forall xyz.(xEy \wedge xEz \rightarrow yEz)$).

Arrow's Theorem

Using our terminology, Arrow's Theorem looks like this [recall $|V| \geqslant 3$]:

Proposition 4 *No nondictatorial, unanimous, grounded and independent aggregator is CR wrt. transitivity and completeness.*

<u>Remarks:</u>

- *CR* wrt. those two properties just means that the output is a well-formed *preference order*.
- Arrow's unanimity is wrt. the strict part of an order. We get the same effect from unanimity + *groundedness*.
- Above holds only for properties wrt. *nonreflexive edges*. We get Arrow's Theorem for weak orders if everyone is reflexive and for linear orders if everyone is irreflexive.
- Btw, *completeness* cannot be characterised by a modal formula, but *connectedness* works just as well.

Similar Results

What sort of results do we get when we move away from properties associated with preferences?

Proposition 5 *No nondictatorial, unanimous, grounded, and independent aggregator is CR wrt. transitivity and seriality.*

Proposition 6 *No nondictatorial, unanimous, grounded, and independent aggregator is CR wrt. the Euclidean property and seriality.*

We prove this using the *ultrafilter technique*. By our lemma, there's a common set of winning coalitions $\mathcal{W}$ for each edge. We get:

- $\emptyset \notin \mathcal{W}$ [from groundedness]
- $C_1, C_2 \in \mathcal{W} \Rightarrow C_1 \cap C_2 \in \mathcal{W}$ [from transitivity, Euclidean, ...]
- $C \in \mathcal{W}$ or $\mathcal{N} \setminus C \in \mathcal{W}$ [from completeness, seriality, ...]

That is, $\mathcal{W}$ is an ultrafilter on $\mathcal{N}$. Thus: $\mathcal{N}$ finite $\Rightarrow$ dictatorial. ✓

Collective Rationality wrt. a Modal Formula

In modal logic, suppose agents agree on the set of worlds V and the valuation function, but provide different accessibility relations.

An aggregator F is *collectively rational wrt. a formula* φ if the following holds for every valuation $Val : \Phi \to 2^V$ and every $x \in V$: $\langle F(\boldsymbol{G}), Val \rangle, x \models \varphi$ whenever $\langle G_i, Val \rangle, x \models \varphi$ for all $i \in \mathcal{N}$.

Note: Most natural scenario is $V := 2^\Phi$ and $Val : p \mapsto \{x \mid p \in x\}$.

Possibility Results

Call formulas in NNF without $\Diamond$'s $\Box$-formulas. We get a nice characterisation for CR aggregators in this case:

Proposition 7 *F is CR wrt. all $\Box$-formulas iff F is grounded.*

Define $\Diamond$-formulas accordingly. We only have a sufficient condition:

Proposition 8 *F is CR wrt. all $\Diamond$-formulas if $F(\boldsymbol{G}) = \langle V, E \rangle$ is such that there exists an individual $i^\star \in \mathcal{N}$ with $E \supseteq E_{i^\star}$.*

Last Slide

We have introduced *graph aggregation* as a generalisation of preference aggregation and considered *collective rationality* wrt. —

- frame properties typically used in modal correspondence theory
- the property of satisfying a modal formula in a given world

We have seen some encouraging *possibility results*, e.g.:

- Any unanimous aggregator will be CR wrt. reflexivity.
- Any grounded aggregator will be CR wrt. $\Box$-formulas.

We have gained a better understanding of *impossibilities:*

- Direct link between CR and axioms, e.g.: transitivity and neutrality
- Arrow-style impossibly results for non-preference graph properties

Future work:

- Can we get stronger results for CR wrt. (arbitrary) formulas?
- Do other definitions of CR wrt. formulas make sense?
- Work on graph-specific aggregators (e.g., successor approval rules).