An Extended Temporal Logic Based on Ordered Trees

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Linear Temporal Logic

Syntax and semantics. Time is modelled as a sequence of states. We have the usual propositional connectives (like negation $\neg$ and conjunction $\wedge$) and a number of temporal operators:

$\Diamond\varphi$	*"φ will be true in* $\boxed{\textit{some}}$ *future state"*
$\Box\varphi$	*"φ will be true in* $\boxed{\textit{all}}$ *future states"*
$\bigcirc\varphi$	*"φ will be true in the* $\boxed{\textit{next}}$ *state"*

Applications. Linear temporal logic has been very successful in the area of systems specification and verification.

But: If we want to *refine* a system (that is, add detail about a subsystem) we may have to revise the entire specification.

▶ We would like to be able to *"zoom in"* . . .

Motivation: The Problem

- Propositional linear temporal logic has been found to be useful for the specification of reactive systems (Pnueli 1977, ...), *but* refining a system specification is problematic (we may have to refine the entire model).

- For many applications, time intervals seem more appropriate than time points, *but* modal interval logics (such as Halpern and Shoham's logic) tend to be undecidable.

Motivation: The Idea

- We propose to *"add a zoom to linear temporal logic"* and obtain a new logic that may be used to represent complex systems evolving over time in a modular fashion (which also facilitates the refinement of specifications).
- Primitive objects in the semantics of this extended temporal logic are "halfway" between *points* and *intervals:* they can be decomposed into smaller units (like time intervals) but they cannot overlap (like time points).
- From an abstract point of view, our logic is best characterised as a *modal logic of ordered trees* (which we call OTL).

Talk Overview

- Ordered Tree Logics
  definition of syntax and semantics
- OTL as a Temporal Representation Language
  time intervals, past and future, properties and events
- Decidability
  brief outline of the proof
- Conclusion
  recap and discussion of future work

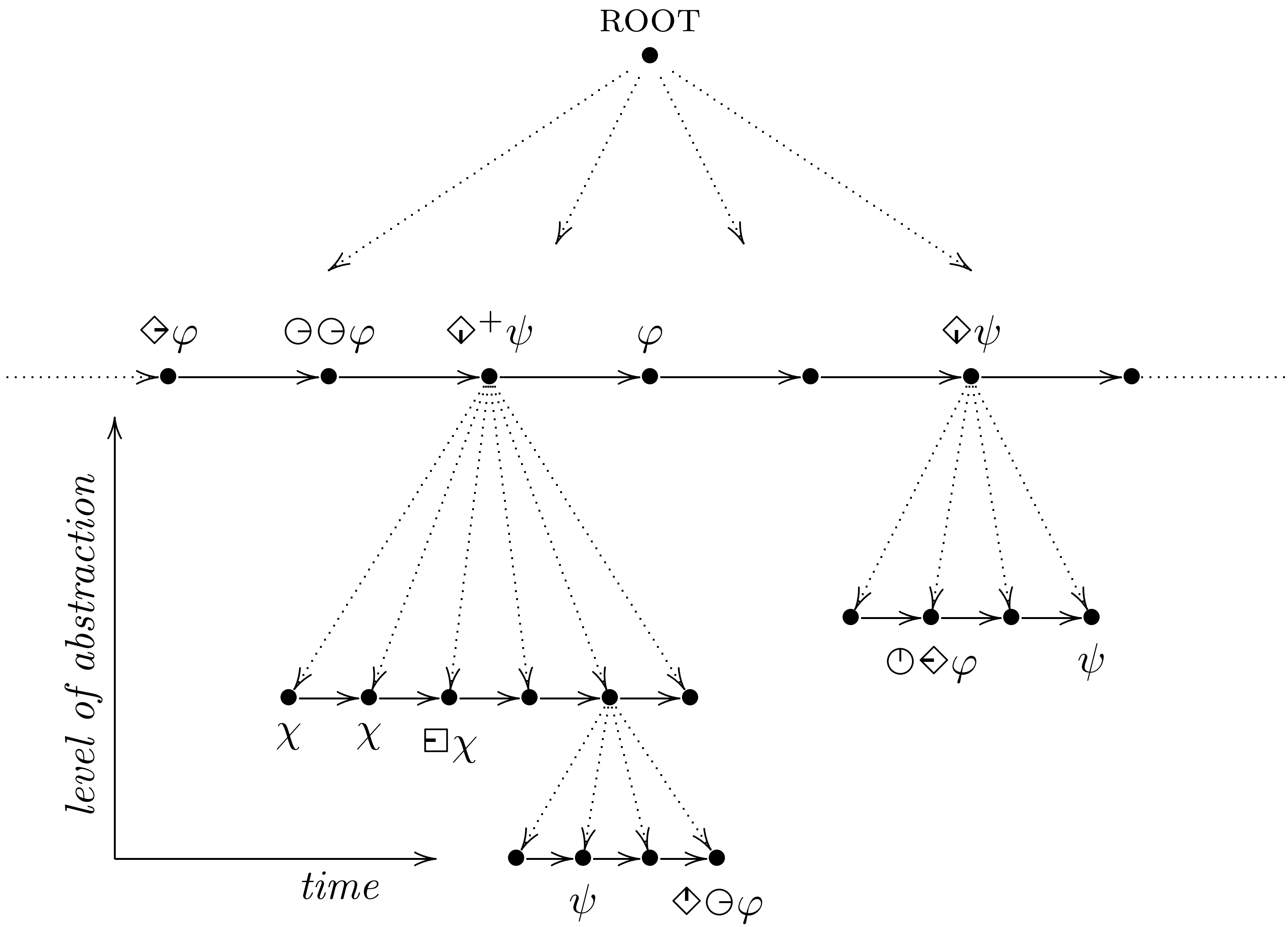
Ordered Tree Logics

Models are based on *ordered trees*. Besides the usual propositional connectives we have a number of modal operators, for example:

$\bigcirc\!\!\!\!\rightarrow\varphi$	*"φ is true at the next righthand sibling (if any)"*
$\Diamond\!\!\!\!\rightarrow\varphi$	*"φ is true at some righthand sibling"*
$\bigcirc\!\!\!\!\uparrow\varphi$	*"φ is true at the parent (if any)"*
$\Diamond\!\!\!\uparrow\varphi$	*"φ is true at some ancestor"*
$\Diamond\!\!\!\downarrow\varphi$	*"φ is true at some child"*
$\Diamond\!\!\!\downarrow^{+}\varphi$	*"φ is true at some descendant"*

We also have modalities ($\bigcirc\!\!\!\!\leftarrow$ and $\Diamond\!\!\!\!\leftarrow$) to refer to *lefthand siblings*. All corresponding *box-operators* are definable, for example:

$$\Box\!\!\!\!\rightarrow\varphi = \neg\Diamond\!\!\!\!\rightarrow\neg\varphi \quad \text{"}\varphi \textit{ is true at all righthand siblings"}$$


ROOT
⬖φ
⊖⊖φ
⬙⁺ψ
φ
⬙ψ
level of abstraction
time
χ
χ
⊟χ
⦶⬖φ
ψ
ψ
⬘⊖φ

Formal Definition of Trees

A *tree* is a pair $\mathcal{T} = (T, R)$, where T is a set and R is an irreflexive binary relation over T satisfying the following conditions:

(1) For every $t \in T$ there exists at most one $t' \in T$ with $(t', t) \in R$.

(2) There exists a unique $r \in T$ such that $\{t \in T \mid (r, t) \in R^*\} = T$.

The elements of T are called *nodes*. The element r from (2) is called the *root* of $\mathcal{T}$. R is called the *child* relation and also gives rise to the following: the *parent* relation R^{-1}, the *descendant* relation R^+, the *ancestor* relation $(R^{-1})^+$, and the *sibling* relation $R^{-1} \circ R$.

We write $[t]_{R^{-1} \circ R} = \{t' \in T \mid (t, t') \in R^{-1} \circ R\}$ for the set of siblings of a node $t \in T$ (including t itself, unless t is the root).

Formal Definition of Ordered Trees

An *ordered tree* is a triple $\mathcal{T} = (T, R, S)$ where (T, R) is a tree, $S \subseteq R^{-1} \circ R$, and $([t]_{R^{-1} \circ R}, S)$ is a strict linear order for every $t \in T$.

If $(t_1, t_2) \in S$ then t_1 is called a *left sibling* of t_2, and t_2 is called a *right sibling* of t_1. If furthermore $(t_1, t_2) \notin S \circ S$ then t_1 is called the *left neighbour* of t_2, and t_2 is called the *right neighbour* of t_1.

Formal Semantics of OTL

Models. A *model* is a pair $\mathcal{M} = (\mathcal{T}, V)$, where $\mathcal{T}$ is an ordered tree and V is a valuation, i.e. a mapping from propositional letters to subsets of the set of nodes in $\mathcal{T}$.

Truth. We inductively define the notion of truth of a formula in a model $\mathcal{M} = (\mathcal{T}, V)$ with $\mathcal{T} = (T, R, S)$ at a node $t \in T$ as follows:

(1) $\mathcal{M}, t \models P$ iff $t \in V(P)$ for propositional letters P;

(2) $\mathcal{M}, t \models \neg\varphi$ iff $\mathcal{M}, t \not\models \varphi$;

(3) $\mathcal{M}, t \models \varphi \wedge \psi$ iff $\mathcal{M}, t \models \varphi$ and $\mathcal{M}, t \models \psi$;

(4) $\mathcal{M}, t \models \bigcirc\!\!\!\!\!\uparrow\;\varphi$ iff t is not the root and $\mathcal{M}, t' \models \varphi$ for the parent t' of t;

(5) $\mathcal{M}, t \models \Diamond\!\!\!\!\uparrow\,\varphi$ iff t has got an ancestor t' with $\mathcal{M}, t' \models \varphi$;

(6) $\mathcal{M}, t \models \bigcirc\!\!\!\!\!\leftarrow\varphi$ iff t has got a left neighbour t' with $\mathcal{M}, t' \models \varphi$;

(7) $\mathcal{M}, t \models \Diamond\!\!\!\!\leftarrow\varphi$ iff t has got a left sibling t' with $\mathcal{M}, t' \models \varphi$;

(8) $\mathcal{M}, t \models \bigcirc\!\!\!\!\!\rightarrow\varphi$ iff t has got a right neighbour t' with $\mathcal{M}, t' \models \varphi$;

(9) $\mathcal{M}, t \models \Diamond\!\!\!\!\rightarrow\varphi$ iff t has got a right sibling t' with $\mathcal{M}, t' \models \varphi$;

(10) $\mathcal{M}, t \models \Diamond\!\!\!\!\downarrow\,\varphi$ iff t has got a child t' with $\mathcal{M}, t' \models \varphi$;

(11) $\mathcal{M}, t \models \Diamond\!\!\!\!\downarrow^{+}\varphi$ iff t has got a descendant t' with $\mathcal{M}, t' \models \varphi$.

Satisfiability and Validity

Satisfiability. A formula φ is *satisfiable* iff there are a model $\mathcal{M}$ and a node t in that model such that $\mathcal{M}, t \models \varphi$.

Global truth. A formula φ is *globally true* in a model $\mathcal{M}$ iff it is true at every node in $\mathcal{M}$. We write $\mathcal{M} \models \varphi$.

Tree validity. A formula φ is called *valid in an ordered tree* $\mathcal{T}$ iff it is true at every node in every model based on $\mathcal{T}$. We write $\mathcal{T} \models \varphi$.

Validity. A formula φ is called *valid* iff it is true at every node in every model. We write $\models \varphi$.

Examples

- A node t is the root of the tree iff ROOT is true at t:

$$\text{ROOT} \quad = \quad \square_{\uparrow}\bot$$

- Similar formulas identify leftmost and rightmost siblings:

$$\text{LEFTMOST} \quad = \quad \square_{\leftarrow}\bot \qquad\qquad \text{RIGHTMOST} \quad = \quad \square_{\rightarrow}\bot$$

- An ordered tree $\mathcal{T}$ is a binary tree iff BINARY is valid in $\mathcal{T}$ (or, equivalently, iff BINARY is globally true in a model based on $\mathcal{T}$):

$$\text{BINARY} \quad = \quad (\text{LEFTMOST} \leftrightarrow \text{RIGHTMOST}) \rightarrow \text{ROOT}$$

- An ordered tree $\mathcal{T}$ is "discretely branching" iff the formula DISCRETE is valid in $\mathcal{T}$:

$$\text{DISCRETE} \quad = \quad \square_{\rightarrow}(\square_{\rightarrow}A \rightarrow A) \rightarrow (\Diamond_{\leftarrow}\square_{\rightarrow}A \rightarrow \square_{\rightarrow}A)$$

This is the standard axiom schema familiar from temporal logic to characterise discrete flows of time.

OTL as a Temporal Representation Language

Temporal interpretation. We can interpret OTL as a (restricted) temporal interval logic as follows:

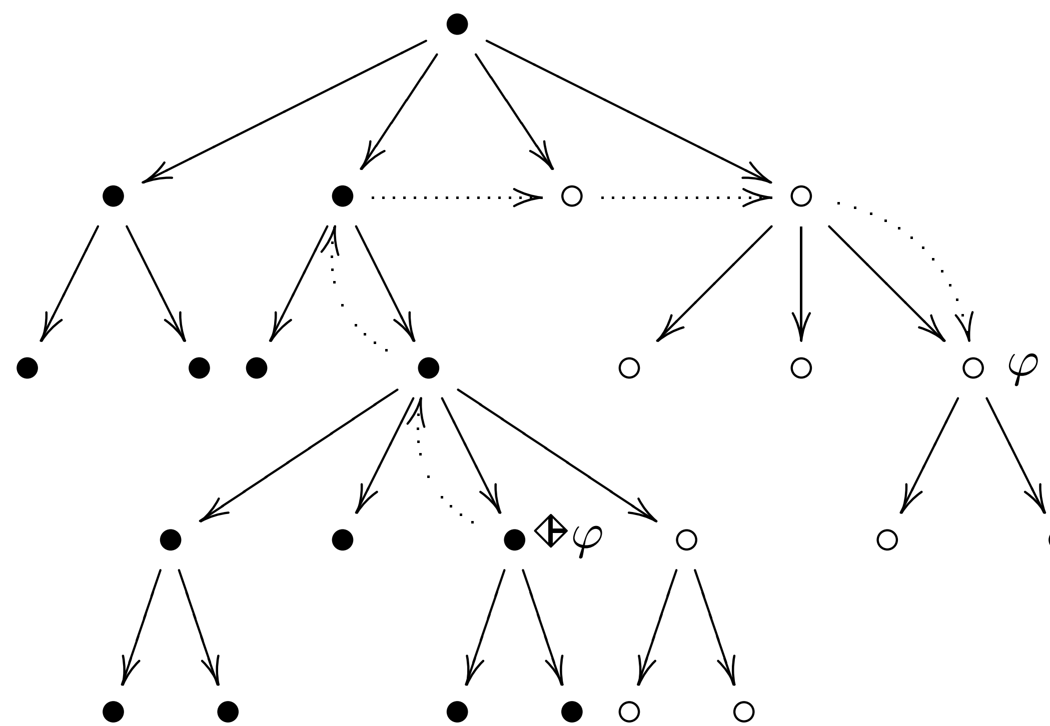
- *nodes* in a tree represent *time intervals*;
- *descendants* represent *subintervals*; and
- the *order declared over siblings* represents an *earlier-later ordering* over time intervals.

Questions. But this raises some questions:

- What is the meaning of, say, the $\Diamond\!\!\!\!\rightarrow$-operator?
  Is it a proper future modality?
- Are models where, say, φ is true at some node t but not at all of t's children meaningful under this temporal interpretation?

Past and Future

If you are allowed to move *up* before and *down* after moving to the right, you can reach *all* the nodes to the right.



Let $\Diamond_{\uparrow}^{*}\varphi = \varphi \vee \Diamond_{\uparrow}\varphi$ and $\Diamond_{\downarrow}^{*}\varphi = \varphi \vee \Diamond_{\downarrow}^{+}\varphi$.
We can now define a *global future modality* as follows:

$$\Diamond_{\updownarrow\rightarrow}\varphi \quad = \quad \Diamond_{\uparrow}^{*}\Diamond_{\rightarrow}\Diamond_{\downarrow}^{*}\varphi$$

Ontological Considerations

Consider the following two basic propositions:

(1) *The sun is shining.*

(2) *I move the pen from the table onto the OHP.*

Propositions like (1) are sometimes called *properties*; propositions like (2) are sometimes called *events* (Allen, 1984).

Properties

Properties like *"The sun is shining."* are *homogeneous* propositions (Shoham, 1987), which we can capture in OTL as follows:

$$
\begin{aligned}
\textsc{downward-hereditary}(\varphi) &= \varphi \rightarrow \boxdot^{+}\varphi \\
\textsc{upward-hereditary}(\varphi) &= \Diamond\top \rightarrow (\boxdot^{+}\varphi \rightarrow \varphi) \\
\textsc{homogeneous}(\varphi) &= \textsc{downward-hereditary}(\varphi) \wedge \\
&\quad\ \textsc{upward-hereditary}(\varphi)
\end{aligned}
$$

Then φ is a homogeneous proposition (with respect to a given model $\mathcal{M}$) iff $\textsc{homogeneous}(\varphi)$ is globally true in $\mathcal{M}$.

Events

Events like *"I move the pen from the table onto the OHP."* may be characterised as propositions that cannot be true at two intervals one of which contains the other.

Shoham (1987) calls such propositions *gestalt:*

$$\textsc{gestalt}(\varphi) \;=\; \varphi \;\rightarrow\; (\overline{\Box}\neg\varphi \;\wedge\; \underline{\Box}^{+}\neg\varphi)$$

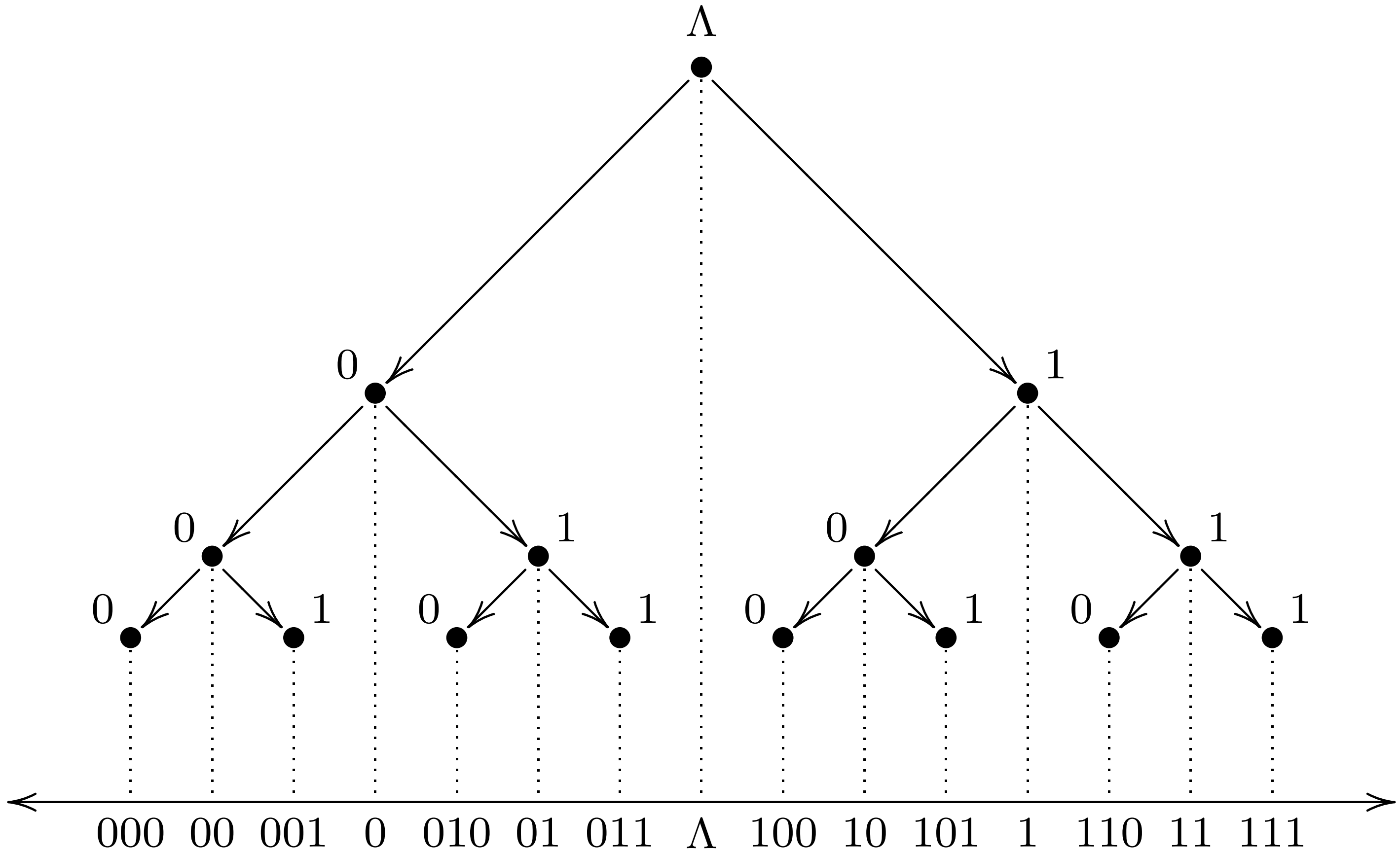
Decidability

Theorem: *OTL is decidable.*

Proof by reduction to Rabin's Theorem (1969) on the decidability of the monadic second-order theory of n successor functions:

- Embed infinite densely ordered tree into S4S tree:
  - encode rationals using two successor functions
  - alternate the two pairs of successor functions to encode tree
- Define OTL accessibility relations; characterise initial subtrees; translate formulas ⇒ decidability for countable ordered trees.
- *Countable Model Lemma:* Show that restriction to countable trees does not matter, by constructing a model based on a countable ordered tree from an arbitrary model.

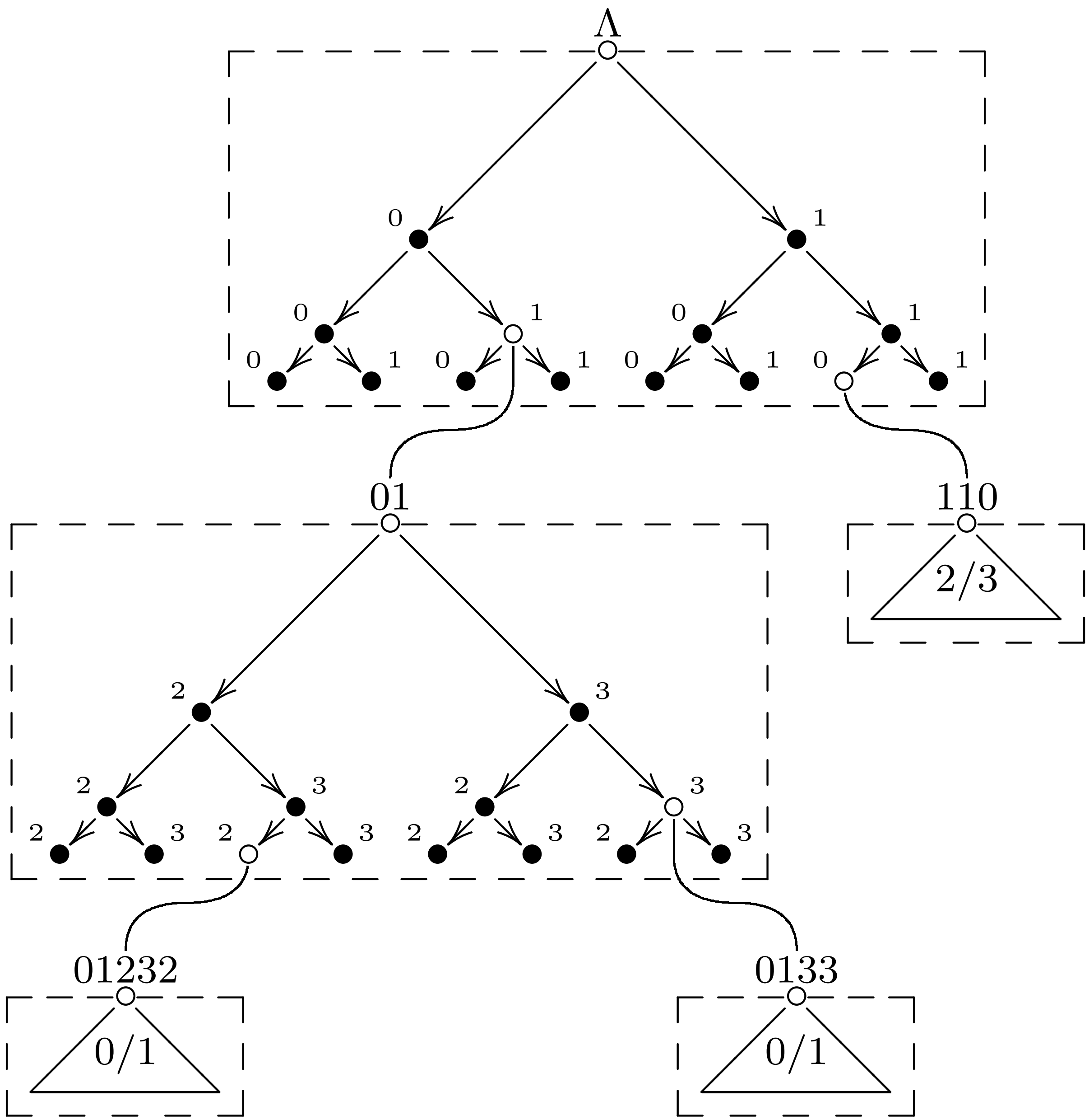
Encoding the Rationals in S2S



We can define an earlier-later ordering as follows:

$$x.1 \leq y \;\vee\; y.0 \leq x \;\vee\; (\exists z)[z.0 \leq x \wedge z.1 \leq y]$$

Encoding Ordered Trees in S4S



Conclusion

- We have introduced a simple yet expressive modal logic for talking about ordered trees.
- Original motivation: linear temporal logic + zoom
- Compromise between point- and interval-based temporal logics:
  - can model subintervals, but not overlapping intervals
  - decidable (unlike many interval logics)
- Also decidability-wise OTL appears to be an interesting borderline case: 2-dimensional modal logics with interacting modalities are often undecidable.

Future Work

- Show decidability of OTL by a direct argument:
  - basic idea would be to use techniques inspired by work on upper complexity bounds of PLTL by Sistla and Clarke to prove a small model theorem
  - various details to be worked out
  - more difficult for extended OTL (with *until*-style operators)
- Devise a Tableaux-based decision procedure
- Applications to Agent Communication Languages:
  - use OTL to represent (nested) communication protocols
  - possibly use model checking to verify protocol conformance

Further Information

For a brief introduction to OTL see:

- U. Endriss and D. Gabbay. Halfway between Points and Intervals: A Temporal Logic Based on Ordered Trees. In *Proceedings of the ESSLLI Workshop on Interval Temporal Logics and Duration Calculi*, Vienna, August 2003. Available soon at `http://www.doc.ic.ac.uk/~ue/pubs/`.

For the full story consult:

- U. Endriss. *Modal Logics of Ordered Trees*. PhD thesis, King's College London, Dept. of Computer Science, January 2003. Available at `http://www.doc.ic.ac.uk/~ue/phd/`.