An Extended Temporal Logic Based on Ordered Trees

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Linear Temporal Logic

Syntax and semantics. Time is modelled as a sequence of states. We have the usual propositional connectives (like negation \( \neg \) and conjunction \( \wedge \)) and a number of temporal operators:

- \( \Diamond \varphi \) “\( \varphi \) will be true in some future state”
- \( \Box \varphi \) “\( \varphi \) will be true in all future states”
- \( \circ \varphi \) “\( \varphi \) will be true in the next state”

Applications. Linear temporal logic has been very successful in the area of systems specification and verification.

But: If we want to refine a system (that is, add detail about a subsystem) we may have to revise the entire specification.

- We would like to be able to “zoom in” …
Motivation: The Problem

- Propositional linear temporal logic has been found to be useful for the specification of reactive systems (Pnueli 1977, ...), but refining a system specification is problematic (we may have to refine the entire model).

- For many applications, time intervals seem more appropriate than time points, but modal interval logics (such as Halpern and Shoham’s logic) tend to be undecidable.
Motivation: The Idea

• We propose to “add a zoom to linear temporal logic” and obtain a new logic that may be used to represent complex systems evolving over time in a modular fashion (which also facilitates the refinement of specifications).

• Primitive objects in the semantics of this extended temporal logic are “halfway” between points and intervals: they can be decomposed into smaller units (like time intervals) but they cannot overlap (like time points).

• From an abstract point of view, our logic is best characterised as a modal logic of ordered trees (which we call OTL).
Talk Overview

• Ordered Tree Logics
  *definition of syntax and semantics*

• OTL as a Temporal Representation Language
  *time intervals, past and future, properties and events*

• Decidability
  *brief outline of the proof*

• Conclusion
  *recap and discussion of future work*
Ordered Tree Logics

Models are based on ordered trees. Besides the usual propositional connectives we have a number of modal operators, for example:

- $\ominus \varphi$ “$\varphi$ is true at the next righthand sibling (if any)”
- $\Diamond \varphi$ “$\varphi$ is true at some righthand sibling”
- $\Box \varphi$ “$\varphi$ is true at the parent (if any)”
- $\Diamond \varphi$ “$\varphi$ is true at some ancestor”
- $\Diamond \varphi$ “$\varphi$ is true at some child”
- $\Diamond^+ \varphi$ “$\varphi$ is true at some descendant”

We also have modalities ($\ominus$ and $\Diamond$) to refer to lefthand siblings. All corresponding box-operators are definable, for example:

$$\Box \varphi = \neg \Diamond \neg \varphi$$ “$\varphi$ is true at all righthand siblings”
Formal Definition of Trees

A tree is a pair $T = (T, R)$, where $T$ is a set and $R$ is an irreflexive binary relation over $T$ satisfying the following conditions:

1. For every $t \in T$ there exists at most one $t' \in T$ with $(t', t) \in R$.
2. There exists a unique $r \in T$ such that \{ $t \in T \mid (r, t) \in R^*$ \} = T.

The elements of $T$ are called nodes. The element $r$ from (2) is called the root of $T$. $R$ is called the child relation and also gives rise to the following: the parent relation $R^{-1}$, the descendant relation $R^+$, the ancestor relation $(R^{-1})^+$, and the sibling relation $R^{-1} \circ R$.

We write $[t]_{R^{-1} \circ R} = \{ t' \in T \mid (t, t') \in R^{-1} \circ R \}$ for the set of siblings of a node $t \in T$ (including $t$ itself, unless $t$ is the root).
Formal Definition of Ordered Trees

An ordered tree is a triple $T = (T, R, S)$ where $(T, R)$ is a tree, $S \subseteq R^{-1} \circ R$, and $([t]_{R^{-1} \circ R}, S)$ is a strict linear order for every $t \in T$.

If $(t_1, t_2) \in S$ then $t_1$ is called a left sibling of $t_2$, and $t_2$ is called a right sibling of $t_1$. If furthermore $(t_1, t_2) \not\in S \circ S$ then $t_1$ is called the left neighbour of $t_2$, and $t_2$ is called the right neighbour of $t_1$. 
Formal Semantics of OTL

Models. A model is a pair $\mathcal{M} = (\mathcal{T}, V)$, where $\mathcal{T}$ is an ordered tree and $V$ is a valuation, i.e. a mapping from propositional letters to subsets of the set of nodes in $\mathcal{T}$.

Truth. We inductively define the notion of truth of a formula in a model $\mathcal{M} = (\mathcal{T}, V)$ with $\mathcal{T} = (T, R, S)$ at a node $t \in T$ as follows:

1. $\mathcal{M}, t \models P$ iff $t \in V(P)$ for propositional letters $P$;
2. $\mathcal{M}, t \models \neg \varphi$ iff $\mathcal{M}, t \nvDash \varphi$;
3. $\mathcal{M}, t \models \varphi \land \psi$ iff $\mathcal{M}, t \models \varphi$ and $\mathcal{M}, t \models \psi$;
4. $\mathcal{M}, t \models \Diamond \varphi$ iff $t$ is not the root and $\mathcal{M}, t' \models \varphi$ for the parent $t'$ of $t$;
5. $\mathcal{M}, t \models \Diamond \varphi$ iff $t$ has got an ancestor $t'$ with $\mathcal{M}, t' \models \varphi$;
6. $\mathcal{M}, t \models \Diamond \varphi$ iff $t$ has got a left neighbour $t'$ with $\mathcal{M}, t' \models \varphi$;
7. $\mathcal{M}, t \models \Diamond \varphi$ iff $t$ has got a left sibling $t'$ with $\mathcal{M}, t' \models \varphi$;
8. $\mathcal{M}, t \models \Diamond \varphi$ iff $t$ has got a right neighbour $t'$ with $\mathcal{M}, t' \models \varphi$;
9. $\mathcal{M}, t \models \Diamond \varphi$ iff $t$ has got a right sibling $t'$ with $\mathcal{M}, t' \models \varphi$;
10. $\mathcal{M}, t \models \Diamond \varphi$ iff $t$ has got a child $t'$ with $\mathcal{M}, t' \models \varphi$;
11. $\mathcal{M}, t \models \Diamond^+ \varphi$ iff $t$ has got a descendant $t'$ with $\mathcal{M}, t' \models \varphi$. 
Satisfiability and Validity

**Satisfiability.** A formula $\varphi$ is *satisfiable* iff there are a model $\mathcal{M}$ and a node $t$ in that model such that $\mathcal{M}, t \models \varphi$.

**Global truth.** A formula $\varphi$ is *globally true* in a model $\mathcal{M}$ iff it is true at every node in $\mathcal{M}$. We write $\mathcal{M} \models \varphi$.

**Tree validity.** A formula $\varphi$ is called *valid in an ordered tree* $\mathcal{T}$ iff it is true at every node in every model based on $\mathcal{T}$. We write $\mathcal{T} \models \varphi$.

**Validity.** A formula $\varphi$ is called *valid* iff it is true at every node in every model. We write $\models \varphi$. 
Examples

• A node $t$ is the root of the tree iff $\text{ROOT}$ is true at $t$:
  \[
  \text{ROOT} = \square \bot
  \]

• Similar formulas identify leftmost and rightmost siblings:
  \[
  \text{LEFTMOST} = \square \bot \quad \text{RIGHTMOST} = \square \bot
  \]

• An ordered tree $T$ is a binary tree iff $\text{BINARY}$ is valid in $T$ (or, equivalently, iff $\text{BINARY}$ is globally true in a model based on $T$):
  \[
  \text{BINARY} = (\text{LEFTMOST} \leftrightarrow \text{RIGHTMOST}) \rightarrow \text{ROOT}
  \]

• An ordered tree $T$ is “discretely branching” iff the formula $\text{DISCRETE}$ is valid in $T$:
  \[
  \text{DISCRETE} = \square (\square A \rightarrow A) \rightarrow (\Diamond \square A \rightarrow \square A)
  \]
  This is the standard axiom schema familiar from temporal logic to characterise discrete flows of time.
OTL as a Temporal Representation Language

**Temporal interpretation.** We can interpret OTL as a (restricted) temporal interval logic as follows:

- *nodes* in a tree represent *time intervals*;
- *descendants* represent *subintervals*; and
- the *order declared over siblings* represents an *earlier-later ordering* over time intervals.

**Questions.** But this raises some questions:

- What is the meaning of, say, the $\Diamond$-operator? Is it a proper future modality?
- Are models where, say, $\varphi$ is true at some node $t$ but not at all of $t$’s children meaningful under this temporal interpretation?
Past and Future

If you are allowed to move up before and down after moving to the right, you can reach all the nodes to the right.

Let $\lozenge^* \varphi = \varphi \lor \lozenge \varphi$ and $\lozenge^* \varphi = \varphi \lor \lozenge^+ \varphi$.

We can now define a global future modality as follows:

$$\lozenge \varphi = \lozenge^* \lozenge \lozenge^* \varphi$$
Ontological Considerations

Consider the following two basic propositions:

(1) *The sun is shining.*

(2) *I move the pen from the table onto the OHP."

Propositions like (1) are sometimes called *properties*; propositions like (2) are sometimes called *events* (Allen, 1984).
Properties

Properties like “The sun is shining.” are homogeneous propositions (Shoham, 1987), which we can capture in OTL as follows:

\[
\text{DOWNWARD-HEREDITARY}(\varphi) = \varphi \rightarrow \Box^+ \varphi
\]

\[
\text{UPWARD-HEREDITARY}(\varphi) = \Diamond \top \rightarrow (\Box^+ \varphi \rightarrow \varphi)
\]

\[
\text{HOMOGENEOUS}(\varphi) = \text{DOWNWARD-HEREDITARY}(\varphi) \land \text{UPWARD-HEREDITARY}(\varphi)
\]

Then \( \varphi \) is a homogeneous proposition (with respect to a given model \( M \)) iff \( \text{HOMOGENEOUS}(\varphi) \) is globally true in \( M \).
Events

Events like “I move the pen from the table onto the OHP.” may be characterised as propositions that cannot be true at two intervals one of which contains the other.

Shoham (1987) calls such propositions gestalt:

\[
\text{GESTALT}(\varphi) = \varphi \rightarrow (\square \neg \varphi \land \square^+ \neg \varphi)
\]
Decidability

Theorem: $OTL$ is decidable.

Proof by reduction to Rabin’s Theorem (1969) on the decidability of the monadic second-order theory of $n$ successor functions:

- Embed infinite densely ordered tree into S4S tree:
  - encode rationals using two successor functions
  - alternate the two pairs of successor functions to encode tree

- Define OTL accessibility relations; characterise initial subtrees; translate formulas $\Rightarrow$ decidability for countable ordered trees.

- Countable Model Lemma: Show that restriction to countable trees does not matter, by constructing a model based on a countable ordered tree from an arbitrary model.
We can define an earlier-later ordering as follows:

\[ x.1 \leq y \lor y.0 \leq x \lor (\exists z)[z.0 \leq x \land z.1 \leq y] \]
An Extended Temporal Logic Based on Ordered Trees

Encoding Ordered Trees in S4S

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Conclusion

• We have introduced a simple yet expressive modal logic for talking about ordered trees.

• Original motivation: linear temporal logic + zoom

• Compromise between point- and interval-based temporal logics:
  – can model subintervals, but not overlapping intervals
  – decidable (unlike many interval logics)

• Also decidability-wise OTL appears to be an interesting borderline case: 2-dimensional modal logics with interacting modalities are often undecidable.
Future Work

• Show decidability of OTL by a direct argument:
  – basic idea would be to use techniques inspired by work on upper complexity bounds of PLTL by Sistla and Clarke to prove a small model theorem
  – various details to be worked out
  – more difficult for extended OTL (with until-style operators)

• Devise a Tableaux-based decision procedure

• Applications to Agent Communication Languages:
  – use OTL to represent (nested) communication protocols
  – possibly use model checking to verify protocol conformance
Further Information

For a brief introduction to OTL see:


For the full story consult: