Collective Rationality in Graph Aggregation

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Talk Outline

I will discuss how to aggregate the information inherent in several directed graphs, using the methodology of social choice theory.

- The Model: Graph Aggregation
- Main Concept: Collective Rationality wrt Graph Properties
- Axioms for Aggregators and Basic Results
- General Impossibility Theorem: Proof and Applications
- Graph Aggregation and Modal Logic

Except for the material on modal logic, this has been published here:

U. Endriss and U. Grandi. Collective Rationality in Graph Aggregation. Proc. 21st European Conference on Artificial Intelligence (ECAI-2014).

Graph Aggregation

Fix a finite set of *vertices* V. A (directed) *graph* $G = \langle V, E \rangle$ based on V is defined by a set of *edges* $E \subseteq V \times V$ (thus: graph = edge-set).

Everyone in a finite group of *agents* $\mathcal{N} = \{1, \ldots, n\}$ provides a graph, giving rise to a *profile* $\boldsymbol{E} = (E_1, \ldots, E_n)$.

An *aggregator* is a function mapping profiles to collective graphs:

$$F : (2^{V \times V})^n \to 2^{V \times V}$$

Examples for aggregators:

- *majority rule:* accept an edge *iff* $> \frac{n}{2}$ of the individuals do
- *intersection rule:* return $E_1 \cap \cdots \cap E_n$

Examples

You may need to use graph aggregation in some of these situations:

- *Elections:* aggregation of preference relations
- *Consensus clustering:* aggregating outputs (equivalence classes) generated by different clustering algorithms
- Aggregation of Dungian *abstract argumentation frameworks* (graphs of attack relations between arguments)
- *Social network analysis:* aggregating influence networks
- *Epistemology:* aggregating Kripke frames for epistemic logics
  - aggregation by intersection = distributed knowledge
  - aggregation by union = shared knowledge
  - aggregation by transitive closure of union = common knowledge

Collective Rationality

Examples for typical properties a graph may or may not possess:

Reflexivity	$\forall x.xEx$
Symmetry	$\forall xy.(xEy \to yEx)$
Transitivity	$\forall xyz.(xEy \wedge yEz \to xEz)$
Seriality	$\forall x.\exists y.xEy$
Completeness	$\forall xy.[x \neq y \to (xEy \vee yEx)]$
Connectedness	$\forall xyz.[xEy \wedge xEz \to (yEz \vee zEy)]$

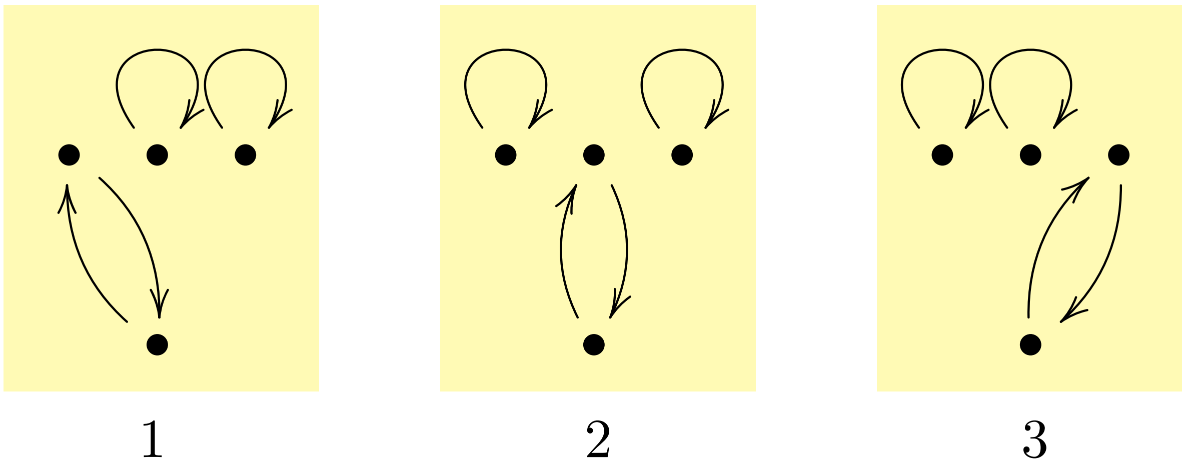
Aggregator F is *collectively rational* (CR) for graph property P if, whenever all individual graphs E_i satisfy P, so does the outcome $F(\boldsymbol{E})$.

▶ Which aggregegatirs are CR for which graph properties?

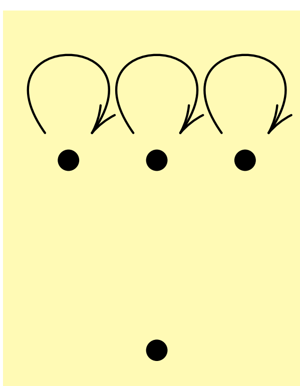
Remark: Same question is studied in *preference aggregation* (CR wrt transitivity) and *judgment aggregation* (CR wrt logical consistency).

Example

Three agents each provide a graph on the same set of four vertices:



If we aggregate using the *majority rule*, we obtain this graph:



Observations:

- Majority rule not collectively rational for *seriality*.
- But *symmetry* is preserved.
- So is *reflexivity* (easy: individuals violate it).

Axioms

Want to study collective rationality for *classes* of aggregators rather than *specific* aggregators (such as the majority rule).

We may want to impose certain *axioms* on $F : (2^{V \times V})^n \to 2^{V \times V}$, e.g.:

- *Anonymous:* $F(E_1, \ldots, E_n) = F(E_{\sigma(1)}, \ldots, E_{\sigma(n)})$
- *Nondictatorial:* for no $i^\star \in \mathcal{N}$ you always get $F(\boldsymbol{E}) = E_{i^\star}$
- *Unanimous:* $F(\boldsymbol{E}) \supseteq E_1 \cap \cdots \cap E_n$
- *Grounded:* $F(\boldsymbol{E}) \subseteq E_1 \cup \cdots \cup E_n$
- *Neutral:* $N_e^{\boldsymbol{E}} = N_{e'}^{\boldsymbol{E}}$ implies $e \in F(\boldsymbol{E}) \Leftrightarrow e' \in F(\boldsymbol{E})$
- *Independent:* $N_e^{\boldsymbol{E}} = N_e^{\boldsymbol{E}'}$ implies $e \in F(\boldsymbol{E}) \Leftrightarrow e \in F(\boldsymbol{E}')$

For technical reasons, we'll restrict some axioms to *nonreflexive edges* $(x, y) \in V \times V$ with $x \neq y$ (NR-neutral, NR-nondictatorial).

Notation: $N_e^{\boldsymbol{E}} = \{i \in \mathcal{N} \mid e \in E_i\}$ = *coalition* accepting edge e in $\boldsymbol{E}$

Basic Results

Proposition 1 *Every unanimous aggregator is CR for reflexivity.*

<u>Proof:</u> If every individual graph includes edge (x, x), then unanimity ensures the same for the collective outcome graph. ✓

Proposition 2 *Every grounded aggregator is CR for irreflexivity.*

<u>Proof:</u> Similar. ✓

Proposition 3 *Every neutral aggregator is CR for symmetry.*

<u>Proof:</u> If the input is not symmetric, we are done. So suppose it is. Thus, (x, y) and (y, x) must have the same support. Then, by CR, either both or neither will get accepted. ✓

Arrow's Theorem

Our formulation in graph aggregation:

> *For $|V| \geqslant 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for ~~reflexivity~~, transitivity and completeness.*

This implies the standard formulation, because:

- weak preference orders = reflexive, transitive, complete graphs
- (weak) Pareto + CR $\Rightarrow$ unanimous + grounded
- nondictatorial = NR-nondictatorial for reflexive graphs
- CR for reflexivity is vacuous (implied by unanimity)

We wanted to know:

- ▶ For what other classes of graphs does this go through?

Our General Impossibility Theorem

Our main result:

> *For $|V| \geqslant 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for any graph property that is contagious, implicative and disjunctive.*

where:

- Implicative $\approx [\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \wedge e_2 \rightarrow e_3]$
- Disjunctive $\approx [\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \vee e_2]$
- Contagious $\approx$ for every accepted edge, there are some conditions under which also one of its "neighbouring" edges is accepted

Examples:

- *Transitivity* is contagious and implicative
- *Completeness* is disjunctive

$\Big\} \Rightarrow$ *Arrow's Theorem*

- *Connectedness* $[xEy \wedge xEz \rightarrow (yEz \vee zEy)]$ has all 3 properties

Winning Coalitions

If an aggregator F is *independent*, then for every edge e there exists a set of *winning coalitions* $\mathcal{W}_e \subseteq 2^{\mathcal{N}}$ such that $e \in F(\boldsymbol{E}) \Leftrightarrow N_e^{\boldsymbol{E}} \in \mathcal{W}_e$.

Furthermore:

- If F is *unanimous*, then $\mathcal{N} \in \mathcal{W}_e$ for all edges e.
- If F is *grounded*, then $\emptyset \notin \mathcal{W}_e$ for all edges e.
- If F is *neutral*, then there is one $\mathcal{W}$ with $\mathcal{W} = \mathcal{W}_e$ for all edges e.

Proof Plan

Given: Arrovian aggregator F (*unanimous*, *grounded*, *independent*)
Want: Impossibility for *collective rationality* for graph property P

This will work if P is *contagious*, *implicative*, and *disjunctive*.

Lemma: CR for *contagious* $P \Rightarrow F$ is NR-*neutral*.

$\Rightarrow F$ characterised by some $\mathcal{W}$: $(x, y) \in F(\boldsymbol{E}) \Leftrightarrow N^{\boldsymbol{E}}_{(x,y)} \in \mathcal{W}$ $[x \neq y]$

Lemma: CR for *implicative* & *disjunctive* $P \Rightarrow \mathcal{W}$ is an *ultrafilter*, i.e.:

(i) $\emptyset \notin \mathcal{W}$ [this is immediate from groundedness]
(ii) $C_1, C_2 \in \mathcal{W}$ implies $C_1 \cap C_2 \in \mathcal{W}$ (closure under intersection)
(iii) C or $\mathcal{N} \setminus C$ is in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$ (maximality)

$\mathcal{N}$ is *finite* $\Rightarrow \mathcal{W}$ is *principal*: $\exists i^\star \in \mathcal{N}$ s.t. $\mathcal{W} = \{C \in 2^{\mathcal{N}} \mid i^\star \in C\}$
But this just means that $i^\star$ is a dictator: F is NR-*dictatorial*. ✓

Neutrality Lemma

Consider any Arrovian aggregator (unanimous, grounded, independent).

Call a property P *xy/zw-contagious* if there exist sets $S^+, S^- \subseteq V \times V$ s.t. every graph $E \in P$ satisfies $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [xEy \rightarrow zEw]$.

CR for xy/zw-contagious P implies: coalition $C \in \mathcal{W}_{(x,y)} \Rightarrow C \in \mathcal{W}_{(z,w)}$

Call P *contagious* if it satisfies (at least) one of the three conditions below:

(i) P is xy/yz-contagious for all $x, y, z \in V$.
(ii) P is xy/zx-contagious for all $x, y, z \in V$.
(iii) P is xy/xz-contagious and xy/zy-contagious for all $x, y, z \in V$.

Example: *Transitivity* ($[yEz] \rightarrow [xEy \rightarrow xEz]$ and $[zEx] \rightarrow [xEy \rightarrow zEy]$)

Contagiousness allows us to reach every NR edge from every other NR edge. Thus, *CR for contagious P* implies $\mathcal{W}_e = \mathcal{W}_{e'}$ for all NR edges e, e'.

So: *Collective rationality* for a *contagious* property implies NR-*neutrality*.

Ultrafilter Lemma

Let F be *unanimous*, *grounded*, *independent*, NR-*neutral*, and *CR* for P.
So there exists a family of winning coalitions $\mathcal{W}$ s.t. $e \in F(\boldsymbol{E}) \Leftrightarrow N_e^{\boldsymbol{E}} \in \mathcal{W}$.

Show that $\mathcal{W}$ is an *ultrafilter* (under certain assumptions on P):

(ii) *Closure under intersections:* $C_1, C_2 \in \mathcal{W} \Rightarrow C_1 \cap C_2 \in \mathcal{W}$

Call P *implicative* if there exist $S^+, S^- \subseteq V \times V$ and $e_1, e_2, e_3 \in V \times V$ s.t. all graphs $E \in P$ satisfy $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \wedge e_2 \rightarrow e_3]$.

Example: transitivity

CR for implicative $P \Rightarrow$ closure under intersections

Proof: consider profile where C_1 accept e_1, C_2 acc. e_2, $C_1 \cap C_2$ acc. e_3

(iii) *Maximality:* C or $\mathcal{N} \setminus C$ in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$

Call P *disjunctive* if there exist $S^+, S^- \subseteq V \times V$ and $e_1, e_2 \in V \times V$ s.t. all graphs $E \in P$ satisfy $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \vee e_2]$.

Example: completeness

CR for disjunctive $P \Rightarrow$ maximality

Proof: consider profile where C accept e_1, $\mathcal{N} \setminus C$ accept e_2

Instantiating the General Impossibility Theorem

I have sketched a proof for the following theorem:

> *For $|V| \geqslant 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for any graph property that is contagious, implicative and disjunctive.*

Many combinations of properties have our meta-properties:

		C/I/D
Transitivity	$\forall xyz.(xEy \wedge yEz \to xEz)$	$++-$
Right Euclidean	$\forall xyz.(xEy \wedge xEz \to yEz)$	$++-$
Left Euclidean	$\forall xyz.(xEy \wedge zEy \to zEx)$	$++-$
Seriality	$\forall x.\exists y.xEy$	$--+$
Completeness	$\forall xy.[x \neq y \to (xEy \vee yEx)]$	$--+$
Connectedness	$\forall xyz.[xEy \wedge xEz \to (yEz \vee zEy)]$	$+++$
Negative Transitivity	$\forall xyz.[xEy \to (xEz \vee zEy)]$	$+-+$

Application: Preference Aggregation

As an immediate corollary to our theorem, we get *Arrow's Theorem* (both for weak orders and for strict linear orders).

Arrow's Theorem does *not* hold for for *partial-order preferences*, as the *intersection rule* has all the required properties. But:

Theorem 4 (Pini et al., 2009) *Every preference aggregation rule for preorders with maximal elements for three or more alternatives that is Arrovian must be a dictatorship.*

Proof: Preorders are reflexive and transitive. Having a maximal element means that at least one alternative is as good as any other. *Transitivity* is contagious and implicative. Property of existence of a *maximal element* is disjunctive. *Reflexivity* of the input together with unanimity means that any NR-dictator is actually a full dictator. ✓

M.S. Pini, F. Rossi, K.B. Venable, and T. Walsh. Aggregating Partially Ordered Preferences. *Journal of Logic and Computation*, 19(3):475–502, 2009.

Application: Consensus Clustering

Clustering algorithms try to partition data points into clusters.
Output is an *equivalence relation* (equivalent = in same cluster).
Don't want a *trivial* clustering: every point is its own cluster.

Consensus clustering is about finding a compromise between the solutions suggetsed by several algorithms: use aggregation.

Theorem 5 *Every aggregator for nontrivial equivalence relations on three or more points that is Arrovian must be a dictatorship.*

Proof: *Transitivity* is both contagious and implicative, while the *nontriviality* condition is disjunctive (disjunction over all edges). *Reflexivity* of the input together with unanimity means that any NR-dictator is actually a full dictator. ✓

Graph Aggregation and Modal Logic

Idea: think of graphs $\langle V, E\rangle$ as Kripke frames and describe graph properties using modal formulas φ.

Suggests natural hierarchy of collective rationality requirements:

- F is *frame-CR* wrt formula φ if $\langle V, E_i\rangle \models \varphi$ for all $i \in \mathcal{N}$ implies $\langle V, F(\boldsymbol{E})\rangle \models \varphi$. [same as notion of CR used so far]
- F is *model-CR* wrt formula φ if, for every valuation $Val : \Phi \to 2^V$, $\langle\langle V, E_i\rangle, Val\rangle \models \varphi$ for all $i \in \mathcal{N}$ implies $\langle\langle V, F(\boldsymbol{E})\rangle, Val\rangle \models \varphi$.
- F is *world-CR* wrt formula φ if, for every $Val : \Phi \to 2^V$ and world $x \in V$, $\langle\langle V, E_i\rangle, Val\rangle, x \models \varphi$ for all i implies $\langle\langle V, F(\boldsymbol{E})\rangle, Val\rangle, x \models \varphi$.

Proposition 6 *This holds for for all aggregators F and all formulas φ: F is world-CR wrt $\varphi \Rightarrow F$ is model-CR wrt $\varphi \Rightarrow F$ is frame-CR wrt φ.*

Example: *majority rule* is frame-CR but not world-CR wrt $p \to \Diamond p$.

World Collective Rationality

Recall that world-CR is our most demanding CR requirement.
Suppose all formulas are in NNF.

Proposition 7 *Any F for which, for every profile $\boldsymbol{E}$, $F(\boldsymbol{E}) \subseteq E_{i^\star}$ for some agent $i^\star$ is world-CR for all $\Box$-formulas (not including any $\Diamond$'s).*

Proposition 8 *Any F for which, for every profile $\boldsymbol{E}$, $F(\boldsymbol{E}) \supseteq E_{i^\star}$ for some agent $i^\star$ is world-CR for all $\Diamond$-formulas (not including any $\Box$'s).*

Proposition 9 *Any representative-voter rule F is world-CR for all formulas (means: there exists $r : (V \times V)^n \to \mathcal{N}$ s.t. $F(\boldsymbol{E}) = E_{r(\boldsymbol{E})}$).*

The converse would hold as well if formulas were fully expressive, but modal formulas cannot distinguish bisimilar models.

Last Slide

I have introduced a simple framework for *graph aggregation* and then considered the concept of *collective rationality*.

- *impossibility result:* Arrovian aggregation impossible if you require CR wrt a contagious, implicative, disjunctive graph property
- *modal logic* perspective suggests different levels of CR
- *applications* in preference aggregation, abstract argumentation, clustering, social network analysis, epistemology, ...

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