Analysis of Matching Mechanisms via SAT Solving

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The Approach

To prove an impossibility in your favourite area of economic theory ...

- Prove *base case:* theorem holds for some fixed domain size ("k").

  Try to express statement in *propositional logic* (typically requiring a few million clauses). Check satisfiability with a *SAT solver*.

  Ideal: *human-readable proof* from *minimal unsatisfiable subset*

- Prove *inductive step:* claim for "n" implies claim for "$n+1$".

  Requires engagement with axioms just as for a classical proof.
  Proof might be technically demanding, *but* result never surprising!

  Ideal: *general lemma* covering *all axioms* of some given type

S. Chatterjee and A. Sen. Automated Reasoning in Social Choice Theory: Some Remarks. *Mathematics in Computer Science*, 2014.

C. Geist and D. Peters. Computer-Aided Methods for Social Choice Theory. In U. Endriss (ed.), *Trends in Computational Social Choice*. AI Access, 2017.

Talk Outline

My plan for this talk is to demonstrate how this approach can be applied to the axiomatic analysis of matching mechanisms.

- Model: one-to-one matching
- Preservation Theorem for axioms expressed in a formal language
- Approach to proving impossibility theorems via SAT solving
- Application: two impossibility theorems for matching

The Model: One-to-One Matching

Two groups of *agents*: $A_n = \{a_1, \ldots, a_n\}$ and $B_n = \{b_1, \ldots, b_n\}$.
Each agent *ranks* all the agents on the opposite side of the market.
Need *mechanism* to return one-to-one *matching* given such a *profile*.

Examples: job markets, marriage markets, ...

Would like a mechanism with good normative properties (*axioms*):

- *Stability:* no a_i and b_j prefer one another over assigned partners
- *Strategyproofness:* best strategy is to truthfully report preferences
- *Fairness:* (for example) no advantage for one side of the market

Gale-Shapley (1962): stable (✓); strategyproof for left side (✓) but not right side (✗) of the market; unfair advantage for left side (✗).

D. Gale and L.S. Shapley. College Admissions and the Stability of Marriage. *American Mathematical Monthly*, 69:9–15, 1962.

Expressing Axioms in Two-Sorted Logic

Would like to have formal language with clear semantics (i.e., a logic) to express axioms, to be able to get results for entire families of axioms.

First-order logic with *sorts*, one for *profiles* and one for agent *indices*, with these basic ingredients:

- $p \triangleright (i, j)$ — in profile p, agents a_i and b_j will get matched
- $j \succ^{\mathrm{A}}_{p,i} j'$ — in profile p, agent a_i prefers b_j to $b_{j'}$ (also for B)
- $top^{\mathrm{A}}_{p,i} = j$ — in profile p, agent a_i most prefers b_j (also for B)
- $p \sim^{\mathrm{A}}_{i} p'$ — profiles p and p' are a_i-variants (also for B)
- $p \rightleftarrows p'$ — swapping sides in profile p yields profile p'
- $\forall_{\mathrm{P}}$ and $\forall_{\mathrm{N}}$ — universal quantifiers for variables of two sorts

Recall that axioms describe properties of mechanisms. So *truth* of a *sentence* φ in our logic is defined relative to a *mechanism* μ.

Example

$$\forall_{\mathrm{P}} p.\forall_{\mathrm{P}} p'.\forall_{\mathrm{N}} i.\forall_{\mathrm{N}} j.\forall_{\mathrm{N}} j' \,.\, \left[(j \succ^{\mathrm{A}}_{p,i} j' \;\wedge\; p \sim^{\mathrm{A}}_{i} p') \rightarrow \neg(p \triangleright (i,j') \;\wedge\; p' \triangleright (i,j))\right]$$

Another Example

$$\forall_{\mathrm{P}} p.\forall_{\mathrm{N}} i.\forall_{\mathrm{N}} j \,.\, \Big[\, (top^{\mathrm{A}}_{p,i} = j \;\wedge\; top^{\mathrm{B}}_{p,j} = i) \;\rightarrow\; (p \triangleright (i,j)) \,\Big]$$

The Preservation Theorem

Call a mechanism *top-stable* if it always matches all mutual favourites.
Call an axiom *universal* if it can be written in the form $\forall \vec{x}.\varphi(\vec{x})$.

Similar to (one direction of) the Łoś-Tarski Theorem in model theory (about preservation of first-order $\forall_1$-formulas in substructures):

Theorem 1 *Let μ^+ be a top-stable mechanism of dimension n that satisfies a given set Φ of universal axioms. If $n > 1$, then there also exists a top-stable mechanism μ of dimension $n-1$ that satisfies Φ.*

Proof idea: Construct larger profile in which extra agents most prefer each other and are least liked by everybody else.

Corollary: enough to prove impossibility theorems for smallest n!

Counterexample

Preservation Theorem might look trivial. *Doesn't this always hold?*

No: some axioms we can satisfy for large but not for small domains.

Suppose we want to design a mechanism under which at least one agent in each group gets assigned to their most preferred partner:

$$\forall_{\mathrm{P}} p.\exists_{\mathrm{N}} i.\forall_{\mathrm{N}} j.[\,(top^{\mathrm{A}}_{p,i} = j) \;\rightarrow\; (p \triangleright (i,j))\,] \wedge$$

$$\forall_{\mathrm{P}} p.\exists_{\mathrm{N}} j.\forall_{\mathrm{N}} i.[\,(top^{\mathrm{B}}_{p,j} = i) \;\rightarrow\; (p \triangleright (i,j))\,]$$

This is *not* universal! Mechanism exists for $n = 3$ but not for $n = 2$.

Proving Impossibility Theorems

Suppose we want to prove an impossibility theorem of this form:

"*for $n \geq k$, no matching mechanism satisfies all the axioms in Φ*"

Our Preservation Theorem permits us to proceed as follows:

- Check all axioms in Φ indeed are universal axioms.
- Check Φ includes (or implies) top-stability.
- Express all axioms for special case of $n = k$ in propositional CNF.
- Using a SAT solver, confirm that this CNF is unsatisfiable.
- Using an MUS extractor, find a short proof of unsatisfiability.

For example, *stability* for $n = 3$ can be expressed in CNF like this:

$$\bigwedge_{p \in B_3!^3 \times A_3!^3} \bigwedge_{i \in \{1,2,3\}} \bigwedge_{j \in \{1,2,3\}} \bigwedge_{\substack{i' \text{ s.t. } p \text{ has} \\ a_i \succ_{b_j} a_{i'}}} \bigwedge_{\substack{j' \text{ s.t. } p \text{ has} \\ b_j \succ_{a_i} b_{j'}}} \left(\neg x_{p \triangleright (i,j')} \vee \neg x_{p \triangleright (i',j)} \right)$$

Application: A Variant of Roth's Theorem

Recall this classic result:

Theorem 2 (Roth, 1982) *For $n \geq 2$, no matching mechanism for incomplete preferences is both stable and two-way strategyproof.*

Remark: In our model (with complete preferences) true only for $n \geq 3$.

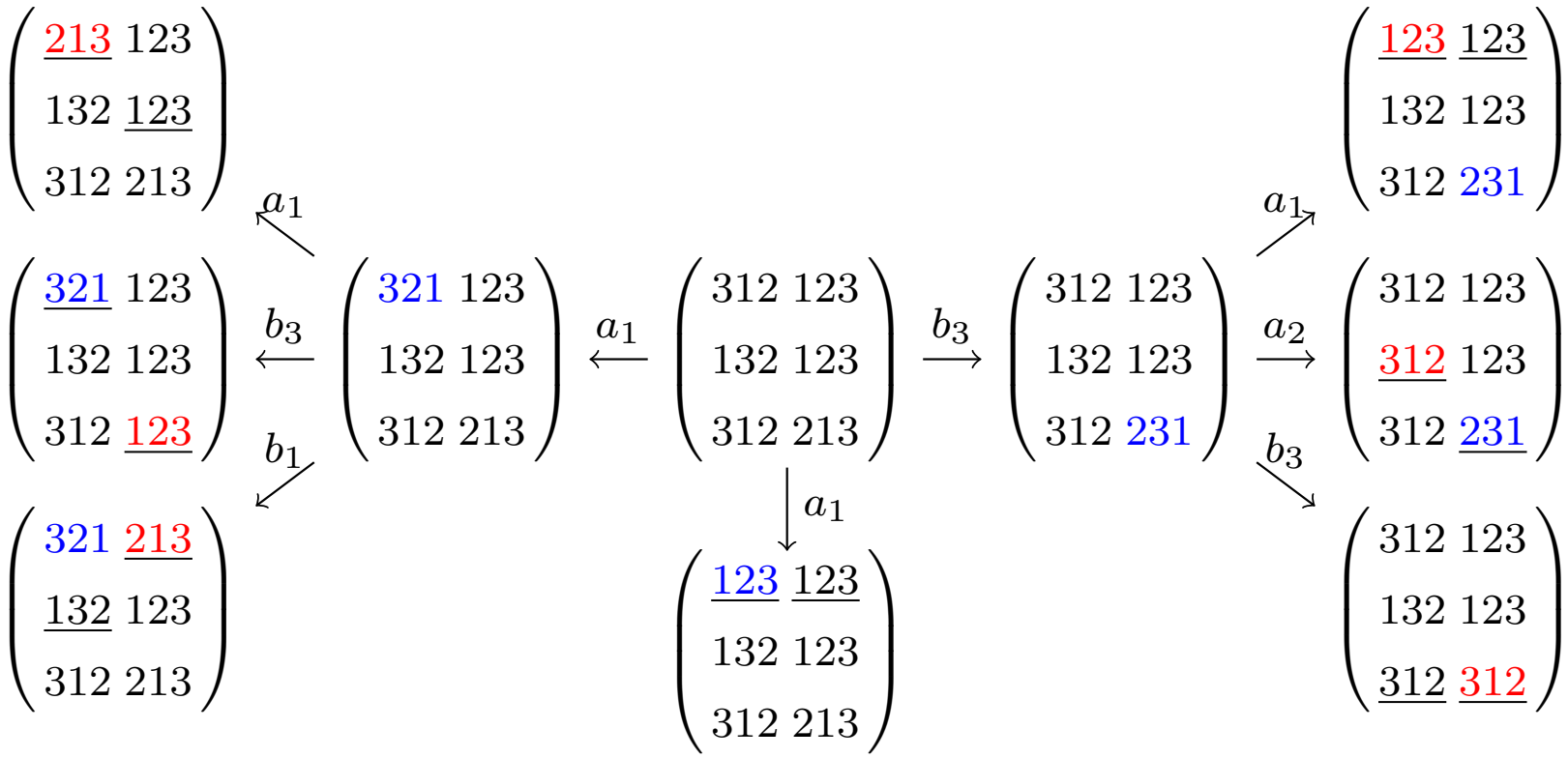
We can use our approach to prove this stronger variant:

Theorem 3 *For $n \geq 3$, no matching mechanism is both top-stable and two-way strategyproof (even in our model).*

By the Preservation Theorem, we are done if the claim holds for $n = 3$. SAT solver says it does, and MUS provides human-readable proof ($\hookrightarrow$).

A.E. Roth. The Economics of Matching: Stability and Incentives. *Mathematics of Operations Research*, 7:617–628, 1982.

Proof of Base Case



Application: Stability vs. Gender-Indifference

Call a mechanism *gender-indifferent* if swapping the two sides of the market ("genders") yields the corresponding swap in the outcome:

$$\forall_{\mathrm{P}} p.\forall_{\mathrm{P}} p'.\forall_{\mathrm{N}} i.\forall_{\mathrm{N}} j\,.\,[\,(p \rightleftarrows p') \;\rightarrow\; (\,p \triangleright (i,j) \;\rightarrow\; p' \triangleright (j,i)\,)\,]$$

Bad news:

Theorem 4 *For $n \geq 3$, there exists no matching mechanism that is both stable and gender-indifferent.*

Here the MUS extractor finds a particularly simple proof: it identifies a "swap-symmetric" profile for which there exists no admissible outcome (two matchings are ruled out by G-I and the other four by stability).

F. Masarani and S.S. Gokturk. On the Existence of Fair Matching Algorithms. *Theory and Decision*, 26(3):305–322, 1989.

Beyond Impossibility Results

What else can we use SAT solving for? Some ideas:

- Satisfiability of a given set of axioms suggests a *possibility result* (but: only for the problem dimension actually tested!).

  Note: Can also be used to prove *independence of axioms*.

- Inspection of models for satisfiable sets of axioms as a method to *design mechanisms* with a good properties (but: huge!).

- Computing *justifications* for outcomes in terms of axioms.

  Example: "Given this profile p, *every* mechanism satisfying all axioms in Φ would assign a_4 to b_2 (and here's why: ...)."

  Corresponds to showing that $\Phi \cup \{\neg(p \triangleright (4,2))\}$ is unsatisfiable. MUS would provide a concrete explanation.

Last Slide

I have shown how to use modern SAT solving technology to largely automate proving impossibility theorems for matching.

Methodological result:

- Provided you are interested in top-stable mechanisms and all your axioms are universal, everything can be automated.

Specific results for one-to-one matching:

- Impossible to get top-stability and two-way strategyproofness.
- Impossible to get stability and gender-indifference.

Future: beyond just one-to-one; other assumptions on preferences.
Also: potential of SAT for economic theory beyond impossibilities.

Paper and full code available here: `tinyurl.com/satmatching`

U. Endriss. Analysis of One-to-One Matching Mechanisms via SAT Solving: Impossibilities for Universal Axioms. Proc. AAAI-2020.