Logic and Social Choice Theory

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Social Choice Theory

SCT studies collective decision making: how should we aggregate the preferences of the members of a group to obtain a “social preference”?

\[
\begin{align*}
\triangle & \succ_1 \bigcirc \succ_1 \square \\
\square & \succ_2 \triangle \succ_2 \bigcirc \\
\bigcirc & \succ_3 \square \succ_3 \triangle \\
\end{align*}
\]

SCT is traditionally studied in Economics and Political Science, but now also by “us”: *Computational Social Choice*. 
Computational Social Choice

Research can be broadly classified along two dimensions —

The kind of social choice problem studied, e.g.:

- electing a winner given individual preferences over candidates
- aggregating individual judgements into a collective verdict
- fairly dividing a cake given individual tastes

The kind of computational technique employed, e.g.:

- algorithm design to implement complex mechanisms
- complexity theory to understand limitations
- logical modelling to fully formalise intuitions
- knowledge representation techniques to compactly model problems
- adaptation for deployment in a multiagent system

Talk Outline

I will briefly introduce four research areas currently active at the ILLC in which we apply logic to social choice theory:

- Logic for the compact representation of large problem instances: social choice in combinatorial domains
- Logic for the formalisation of social choice mechanisms: from the axiomatic method to logics for social choice
- Logic as a basis for the verification and discovery of theorems in social choice theory: automated reasoning for social choice theory
- Logic as the object of aggregation: judgment aggregation and the computational complexity of judgment aggregation
Social Choice in Combinatorial Domains

Many social choice problems have a *combinatorial structure*:

- Elect a *committee* of \( k \) members from amongst \( n \) candidates.
- Find a good *allocation* of \( n \) indivisible goods to agents.

Seemingly small problems generate huge numbers of alternatives:

- Number of 3-member committees from 10 candidates: \( \binom{10}{3} = 120 \)
  (i.e. \( 120! \approx 6.7 \times 10^{198} \) possible rankings)

- Allocating 10 goods to 5 agents: \( 5^{10} = 9765625 \) allocations and \( 2^{10} = 1024 \) bundles for each agent to think about

We need good *languages* for representing preferences!

Ordinal Preferences: CI-Nets

Until recently there has been no compact language for ordinal preferences that are monotonic. \(\leadsto\) *Conditional Importance Networks*:

- A CI-net is a set of CI-statements of the form \(S^+, S^- : S_1 \succ S_2\).
  ("if I own all the items in \(S^+\) and none of those in \(S^-\), then obtaining set \(S_1\) is more important to me than obtaining set \(S_2\)")
- The *preference order* induced by a CI-net is the smallest partial order that is monotonic and satisfies all its CI-statements.

We are also using (simple fragments of) CI-nets to model *fair division*:

- Given a group of agents’ individual preferences over a set of indivisible goods, can we find an allocation that is envy-free?


Cardinal Preferences: Weighted Goals

Weighted goals are a logic-based language for to compactly represent utility functions over binary combinatorial domains.

- Propositional language over $PS$. Want to model $u : 2^{PS} \rightarrow \mathbb{R}$.
- Formulas of $L_{PS}$ represent goals. Weights represent importance.
- For each truth assignment, aggregate weights of satisfied formulas.

Results include:

- **Expressivity**: with sum aggregation, positive goals with positive weights can express all monotonic functions, and only those
- **Complexity**: social welfare maximisation is NP-hard for max aggregation, even if all weighted goals have the form $(p \land q, 1)$


Finer Analysis via Linear Logic

Weighted goals cannot express statements such as this:

“getting $p$ has value 5 to me, but getting $p$ twice has value 8”

But being able to model this is important for combinatorial auctions and negotiation in multiagent systems.

Resource-sensitive logics, in particular linear logic, can speak about the multiplicity of items.

The Axiomatic Method in Social Choice Theory

Modern SCT has always made use of logic, albeit informally. The first example of the “axiomatic method” was *Arrow’s Theorem* (1951):

*Any aggregation mechanism for a finite group of individuals to rank \( \geq 3 \) alternatives that satisfies the weak Pareto condition and independence or irrelevant alternatives must be dictatorial.*

The three axioms involved are:

- **Weak Pareto**: if all individuals prefer \( x \) to \( y \), then so should society
- **IIA**: the social ranking of \( x \) vs. \( y \) should only depend on the individual rankings of \( x \) vs. \( y \)
- **Nondictatoriality**: the aggregator should not be a dictatorship, i.e., a function that just copies the ranking of a fixed individual
Logics for Social Choice Theory

We have shown how to model the Arrovian framework of preference aggregation (PA) in FOL. Arrow’s Theorem reduces to this:

\[ T_{PA} \cup \{\text{PAR}, \text{IIA}, \text{NDIC}\} \text{ does not have a finite model.} \]

This is interesting for (at least) two reasons:

- It tells us something about the nature of the axioms proposed in SCT (e.g., second-order quantification is not needed).
- It can form the basis for the verification of results in SCT using automated theorem provers.

Related work: (new) modal logic (Ågotnes et al., JAAMAS 2010); propositional logic (Tang & Lin, AIJ 2009); HOL (Nipkow, JAR 2009). For the latter two the focus is on automated reasoning.

Automated Discovery of Theorems

Another area of SCT is *ranking sets of objects*: how do you extend a preference order on objects to a preference order on sets of objects?

**Example:** The *Kannai-Peleg Theorem* (JET, 1984) shows that for sets $\mathcal{X}$ with $|\mathcal{X}| \geq 6$ it is impossible to extend total orders $\succeq$ on $\mathcal{X}$ to weak orders $\preceq$ on $2^{\mathcal{X}} \setminus \{\emptyset\}$ in a manner that respects:

- **Dominance:** prefer $A \cup \{x\}$ to $A$ whenever you prefer $x$ to all $y \in A$, and prefer $A$ to $A \cup \{x\}$ whenever you prefer all $y \in A$ to $x$
- **Independence:** weakly prefer $A \cup \{x\}$ to $B \cup \{x\}$ if you (strictly) prefer $A$ to $B$ and $x \notin A \cup B$

Approach to derive similar new results for this domain:

- Use model-theoretic argument to show that for axioms of certain syntactic form, impossibilities established for $|\mathcal{X}| = k$ always generalise.
- Translate small instances into propositional logic and use SAT solver.

Judgment Aggregation

Preferences are not the only structures we may wish to aggregate. JA studies the aggregation of judgments on logically inter-connected propositions. Example:

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$p \rightarrow q$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judge 1:</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Judge 2:</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Judge 3:</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Majority:</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

Paradox: each *individual* judgment set is *consistent*, but the *collective* judgment arrived at using the *majority rule* is not
Complexity of Judgment Aggregation

We have initiated a study of the computational complexity of JA:

- Performing aggregation is polynomial for the *premise-based procedure* and NP-complete for the *distance-based procedure*.
- *Manipulation* is NP-complete for the *premise-based procedure*.
- Deciding *safety of the agenda* (does a given agenda rule out paradoxes?) is \( \Pi_2^P \)-complete for the *majority rule*.

Deciding safety of the agenda wrt. a class of procedures characterised by axioms such as *anonymity*, *neutrality*, and *independence* typically has the same complexity.


Last Slide

- COMSOC is an exciting area of research bringing together ideas from mathematical economics (particularly social choice theory) and computer science (including logic).

- Examples of ongoing research at the ILLC we have reviewed here:
  - Compact Representation of Preferences
  - Logical Modelling of Social Choice Mechanisms
  - Automated Reasoning for Social Choice Theory
  - Judgment Aggregation

- COMSOC website: http://www.illc.uva.nl/COMSOC/