What's in an axiom?

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Computational Social Choice

The field of computational social choice (COMSOC) is concerned with the design and analysis of methods for collective decision making.

Talk Outline

In our work, we often use *axioms* to describe properties of mechanisms. For most such work, what matters are the *specifics* of concrete axioms.

Today, I instead want to talk about the *nature of axioms* in general:

- What is the formal *meaning* of a given axiom?
- Are there natural *classifications* to put order in the space of axioms?

The Model

We focus on irresolute social choice functions for variable electorates.

Terminology

set of *alternatives* = finite set X

preference = linear order on X = element of $\mathcal{L}(X)$

universe = finite set $N^\star$ of *agents*

electorate = set $N \subseteq N^\star$ of agents reporting a preference

profile = function R from some electorate N to $\mathcal{L}(X)$

outcome = nonempty subset of X (ties are allowed)

Now a *voting rule* (or SCF) is a function mapping any given profile in $\textsc{Prof} := \mathcal{L}(X)^{N \subseteq N^\star}$ to an outcome in $\textsc{Out} := 2^X \setminus \{\emptyset\}$:

$$F : \textsc{Prof} \to \textsc{Out}$$

Remark: Much (all?) of what we'll do also works for other models.

Axioms

An axiom is a *normatively desirable property* of voting rules F.

Examples:

- Anonymity = "treat all agents the same"
- Pareto = "do not select dominated alternatives"
- Strategyproofness = "don't incentivise misreporting of preferences"

Usual: Is axiom A normatively *adequate*? Is it *useful* (for the paper)?

Now: What is the *meaning* of axiom A? How do we *define* it?

Example: Defining the Anonymity Axiom

Start with an *intuitive* expression of the idea:

> *The voting rule we use should treat all agents the same.*

Then turn it into a mathematically *rigorous* definition:

> $F(R) = F(\sigma \circ R)$ for all profiles R and permutations $\sigma : N^\star \to N^\star$

And maybe even provide a *formal* definition in a formal language:

$$\bigwedge_{R \in \textsc{Prof}} \; \bigwedge_{\sigma \in S_{N^\star}} \; \bigwedge_{\substack{R' \in \textsc{Prof} \text{ s.t.} \\ R'(i) = R(\sigma(i))}} \; \bigwedge_{x \in X} \; p_{R,x} \to p_{R',x}$$

Or be *explicit* and just point to the set of *all* anonymous rules:

$$\{\,\textsc{Borda}, \textsc{Copeland}, \textsc{Plurality}, \ldots, F_{4711}, \ldots\}$$

Meaning of Axioms

Two ways of fixing the *meaning* of an *axiom* A:

- *intensional* definition: list necessary and sufficient conditions
- *extensional* definition: enumerate voting rules satisfying A

Aside: Distinction goes back to Gottlob Frege (Sinn *vs.* Bedeutung).

The intensional approach is the common one in SCT:

- good for intuitions, close to philosophical starting point
- but methodologically *ad hoc*, no general formalism

So let's try the extensional approach . . .

G. Frege. Über Sinn und Bedeutung. *Zeitschrift für Philosophie und Philosophische Kritik*, 100(1):25–50, 1892.

Extensional Semantics of Axioms

The *interpretation* (or *extension*) of an axiom A is a set of voting rules:

$$\mathbb{I}(A) \quad \subseteq \quad (\textsc{Prof} \to \textsc{Out})$$

such that $F \in \mathbb{I}(A)$ iff *F satisfies A*

Permits unambiguous definition of meaning of any conceivable axiom.

Applications

Let's review some applications of $\mathbb{I}(\cdot)$ as a notational tool:

- Example for a *relationship* between axioms:

  $\mathbb{I}(\textsc{Pareto}) \ \subseteq \ \mathbb{I}(\textsc{Faithfulness})$

- Example for a *characterisation* result:

  $\mathbb{I}(\textsc{Ano}) \cap \mathbb{I}(\textsc{Neu}) \cap \mathbb{I}(\textsc{PosRes}) \ = \ \{\textsc{Majority}\}$ for $|X| = 2$

- Example for an *impossibility* result:

  $\mathbb{I}(\textsc{Res}) \cap \mathbb{I}(\textsc{Onto}) \cap \mathbb{I}(\textsc{SP}) \cap \mathbb{I}(\textsc{NonDict}) \ = \ \emptyset$ for $|X| \geqslant 3$

Classifying Axioms

We now can classify axioms in terms of their *strength*. Like this:

$$strength(A) \;=\; \frac{1}{|\mathbb{I}(A)|}$$

Other classification approaches coming up next:

- Which profiles does an axiom talk about?
- How many profiles at a time does an axiom constrain?

Axioms Talking about Profiles

An axiom may be "talking" about one profile but not another.
Intuitively clear for intensional definitions. But for extensional ones?

Example 1: *Pareto* = "do not select *dominated* alternatives"

talks about	*does not talk about*
$a \succ b \succ c$	$a \succ b \succ c$
$b \succ c \succ a$	$b \succ c \succ a$
$b \succ a \succ c$	$c \succ a \succ b$

Example 2: *Anonymity* = "be invariant under permutations of agents"
talks about *all* profiles (yet fixes the outcome for none!)

Can we provide a general definition for this concept?

Axioms Talking about Profiles

For any axiom A, define $\mathbb{P}(A)$ as the intersection of all sets $S \subseteq \textsc{Prof}$ for which there exists a family $\mathcal{F}_S \subsetneq (S \to \textsc{Out})$ such that:

$$\mathbb{I}(A) \;=\; \mathcal{F}_S \otimes \{F : (\textsc{Prof} \setminus S) \to \textsc{Out}\}$$

We obtain the following "theorem":

Axiom A talks about profile R iff $R \in \mathbb{P}(A)$.

To get the intuition, check these cases:

- $A =$ Pareto $\;\to\;$ $\mathbb{P}(A) = \{R \mid$ some x is dominated in $R\}$
- $A =$ Anonymity $\;\to\;$ $\mathbb{P}(A) = \textsc{Prof}$

Recall: $\mathbb{I}(A) = \{F : \textsc{Prof} \to \textsc{Out} \mid F$ satisfies $A\}$

Intraprofile and Interprofile Axioms

Fishburn was the first (?) to distinguish *intra-* and *interprofile* axioms:

Pareto	Anonymity
Condorcet	Monotonicity
Resoluteness	Reinforcement
⋮	⋮

Clear enough in practice for concrete axioms.

But what about a general definition?

P.C. Fishburn. *The Theory of Social Choice.* Princeton University Press, 1973.

A Hierarchy of Axioms

Call axiom A a *k-axiom* if k is the smallest integer such that:

$$\mathbb{I}(A) \;=\; \bigcap_{(R_1,\ldots,R_k)\,\in\,\textsc{Prof}^k} \{\, F \mid (F(R_1),\ldots,F(R_k)) \in A(R_1,\ldots,R_k) \,\}$$

$$\text{where } A(R_1,\ldots,R_k) \;:=\; \{\, (F'(R_1),\ldots,F'(R_k)) \mid F' \in \mathbb{I}(A) \,\}$$

So a k-axiom only ever imposes a constraint on k profiles at a time.

Some observations:

- Fishburn's intraprofile axioms $=$ 1-axioms
- Fishburn's interpofile axioms $\approx$ k-axioms with $k > 1$ [more soon]
- Every axiom is a k-axiom for some $k \leqslant |\textsc{Prof}|$.

Recall: $\mathbb{I}(A) = \{F : \textsc{Prof} \to \textsc{Out} \mid F \text{ satisfies } A\}$

Active and Passive Intraprofile Axioms

Fishburn further divides intraprofile axioms into those that are *active* (that "involve specific conditions on contents") and *passive* axioms:

Pareto	Resoluteness
Condorcet	
⋮	⋮

We know how to formalise this!

Axiom A is passive only if $\mathbb{P}(A) = \textsc{Prof}$.

P.C. Fishburn. *The Theory of Social Choice.* Princeton University Press, 1973.

Universal and Existential Axioms

Fishburn restricts the terms intra- and interprofile to *universal* axioms, and distinguishes those from *existential* axioms such as this:

Nonimposition = "every $x \in X$ should win alone in *some* profile"

Intuitively, this is about the type of quantification over profiles:

> "existential [axioms] are based primarily on existential qualifiers [. . .] universal [axioms] do not use existential qualifiers in any way, or [. . .] in a secondary manner"

Even less clear what any of this might mean when there is no language.

But: Typical "existential" axioms are k-axioms for $k = |\textsc{Prof}|$.

P.C. Fishburn. *The Theory of Social Choice.* Princeton University Press, 1973.

Last Slide

I shared a few ruminations about the nature of axioms culminating in language-independent definitions of three fundamental concepts:

- the meaning of an axiom
- the notion of an axiom talking about a profile
- the structural complexity of the constraints an axiom can impose

For full details, see Chapter 2 of Marie Schmidtlein's MSc thesis.

M.C. Schmidtlein. *Voting by Axioms*. MSc thesis, University of Amsterdam, 2022.