Logic and Social Choice Theory

Ulle Endriss
Institute for Logic, Language and Computation
University of Amsterdam

Guest Lecture for “Logic, Language and Computation”
Master of Logic, 10 December 2018
Social Choice Theory

*Social choice theory* is about methods for *collective decision making*, such as *political* decision making by groups of *economic* agents.

Its methodology ranges from the *philosophical* to the *mathematical* to the *algorithmic* (the latter in *computational social choice*).

Its findings are relevant to a variety of *applications*, such as these:

- How to choose a president given people’s preferences?
- How to combine the website rankings of multiple search engines?
- How to fairly divide resources between several stake-holders?
- How to assign student doctors to hospitals?
- How to aggregate the views of different judges in a court case?

The most widely studied scenario is that of *preference aggregation*, with $n$ agents each ranking $m$ alternatives.
Outline

- Examples for preference aggregation scenarios and rules
- Example for a classical result: Arrow’s Theorem (with proof)
- Examples for recent (mostly local) work on logic and SCT
The Plurality Rule

Under the *plurality rule*, the alternative ranked first most often wins:

Voter 1: \( x \succ y \succ w \succ z \)
Voter 2: \( x \succ y \succ w \succ z \)
Voter 3: \( y \succ z \succ w \succ x \)
Voter 4: \( z \succ y \succ w \succ x \)

The most widely used rule in practice (by far!). *But is it any good?*
The Borda Rule

Under the *Borda rule*, each voter gives $m - 1$ points to the alternative she ranks first, $m - 2$ to the alternative she ranks second, and so forth:

- **Voter 1:** $x \succ z \succ y$  
  $x : 2 + 2 + 1 + 1 + 0 = 6$
- **Voter 2:** $x \succ z \succ y$  
  $y : 0 + 0 + 2 + 2 + 1 = 5$
- **Voter 3:** $y \succ x \succ z$  
  $z : 1 + 1 + 0 + 0 + 2 = 4$
- **Voter 4:** $y \succ x \succ z$
- **Voter 5:** $z \succ y \succ x$

A clear advantage over the plurality rule is that we use much more of the information present in the profile to come to a decision.

But there still is a problem: *What if $y$ challenges the winner $x$?*
The Condorcet Rule

Under the Condorcet rule, we run one-to-one majority contests between all pairs of alternatives and elect the one that performs best.

Nice idea. But: Do you see the problem with this “definition”? 

Voter 1: \( x \succ y \succ z \)
Voter 2: \( y \succ z \succ x \)
Voter 3: \( z \succ x \succ y \)
The Axiomatic Method

So how do you decide what is the right aggregation rule to use?

The classical approach is to use the *axiomatic method*:

- identify good axioms: normatively appealing high-level properties
- give mathematically rigorous definitions of your axioms
- explore the logical consequences of your definitions
The Model

The voters from a finite set $N = \{1, \ldots, n\}$, with $n \geq 2$, all rank the alternatives in a set $X$ by supplying a strict linear order in $L(X)$.

We are interested in preference aggregation rules of this form:

$$F : L(X)^n \rightarrow L(X)$$

Thus: Given a profile $(\succ_1, \ldots, \succ_n) \in L(X)^n$ of rankings, one for each voter, we want our rule to return a single collective ranking.

Remark: Alternatively, we could study rules $F : L(X)^n \rightarrow X$ to return the “best” alternative. Essentially the same. Do you see why?
**First Axiom: The Pareto Condition**

A preference aggregation rule $F$ satisfies the (weak) *Pareto condition* if it ranks $x$ above $y$ whenever all individual voters do:

$$\{i \in N \mid x \succ_i y\} = N \Rightarrow x \succ y \text{ for } \succ = F(\succ_1, \ldots, \succ_n)$$

**Discussion:** Desirable to use rules that satisfy this axiom?
Second Axiom: Arrow’s Independence Condition

An aggregation rule $F$ satisfies Arrow’s condition of the *independence of irrelevant alternatives* (IIA) if the relative ranking of $x$ and $y$ it returns depends only on the relative rankings of $x$ and $y$ in the profile:

$$\{i \mid x \succ_i y\} = \{i \mid x \succ'_i y\} \Rightarrow F(\succ_1, \ldots, \succ_n) = x ? y \quad F(\succ'_1, \ldots, \succ'_n)$$

**Discussion:** *Desirable to use rules that satisfy this axiom?*
Summary of Desiderata

We would like to find a rule $F : \mathcal{L}(X)^n \to \mathcal{L}(X)$ that satisfies both the Pareto condition and IIA. How about the rules discussed so far?

- Plurality
- Borda
- Condorcet

All of these rules were originally formulated for just finding a winning alternative (not a ranking) and all of them might run into tie-breaking issues. We can handle these. Focus on the bigger picture instead.
**Arrow’s Impossibility Theorem**

A rule $F$ is a *dictatorship* if there exists a “dictator” $i^* \in N$ such that $F(\succ_1, \ldots, \succ_n) = \succ_{i^*}$ for every profile $(\succ_1, \ldots, \succ_n)$. *Bad!*

**Theorem 1 (Arrow, 1951)** Any aggregation rule for $\geq 3$ alternatives that satisfies the Pareto condition and IIA must be a dictatorship.

**Remarks:**

- not true for 2 alternatives (*counterexample?*)
- opposite direction also holds (so this is a characterisation result)
- dictatorial does *not* just mean: outcome = someone’s preference

**Next:** Proof (following Geanakoplos, 2005).

---


**Extremal Lemma**

Assume there are \( \geq 3 \) alternatives and \( F \) satisfies Pareto and IIA.

**Claim:** If all individuals rank \( y \) either top or bottom, then so does \( F \).

**Proof:** Suppose otherwise, i.e., all individuals rank alternative \( y \) either top or bottom, but \( F \) does not. Write \( \succ \) for the ranking returned by \( F \).

1. Then there exist alternatives \( x, z \in X \) such that \( x \succ y \) and \( y \succ z \).
2. By IIA, this does not change if we move \( z \) above \( x \) in every individual ranking (as doing so we don’t cross the extremal \( y \)).
3. By Pareto, in the new profile we must have \( z \succ x \).
4. But we still have \( x \succ y \) and \( y \succ z \), so by transitivity we get \( x \succ z \).

Contradiction. \( \checkmark \)
Existence of an Extremal-Pivotal Voter

Fix some alternative $y$. Call a voter extremal-pivotal if she can push $y$ from the bottom to the top in $F$ for at least one profile.

Claim: There exists an extremal-pivotal voter (for $y$).

Proof: Consider a profile where every voter ranks $y$ at the bottom. By Pareto, so does $F$. Let voters switch $y$ to the top, one by one. By the Extremal Lemma, after each step, $y$ is still extremal in $F$. By Pareto, at the end of this process, $F$ ranks $y$ at the top.

So there must be a point where $y$ jumps from the bottom to the top. The individual making the corresponding switch is extremal-pivotal. ✓

Remark: Only works because $N$ is finite. Do you see why?

Notation: Let $\alpha$ the profile just before the jump and $\beta$ the profile just after the jump. Let $i$ be the extremal-pivotal voter we just found.
**Dictatorship for Most Alternatives**

Recall: \( i \) is extremal-pivotal for \( y \) in \( \alpha \) (\( y \) is at bottom for \( i \) and \( F' \)), from where she can force \( \beta \) (\( y \) is at top for \( i \) and \( F' \)).

Claim: Voter \( i \) can dictate the relative ranking under \( F \) of any two alternatives \( x \) and \( z \) that are different from \( y \).

Proof: W.l.o.g., suppose \( i \) wants to place \( x \) above \( z \).

Let \( i \) vote almost as in \( \beta \), but moving \( x \) to the top: \( x \succ_i y \succ_i z \).

Let all others rank \( y \) as in \( \beta \), but otherwise (including the relative ranking of \( x \) and \( z \)) vote as they wish. Consider the resulting profile \( \gamma \):

- Observe that in \( \gamma \) all relative rankings of \( x \) and \( y \) are as in \( \alpha \).
  - So by \( IIA \), we must still have \( x \succ y \).
- Observe that in \( \gamma \) all relative rankings of \( y \) and \( z \) are as in \( \beta \).
  - So by \( IIA \), we must still have \( y \succ z \).

By transitivity, we get \( x \succ z \). By \( IIA \), this continues to hold if others change their relative rankings of alternatives other than \( x \) and \( z \).  

✓
Dictatorship for All Alternatives

Recall: Voter $i$ can dictate relative rankings of $x$ and $z$ (anything $\neq y$).

Claim: Voter $i$ can also dictate relative rankings involving $y$.

Proof: We can use a similar construction as before to show:

- Some voter $j$ can dictate relative rankings of $x$ and $y$ ($\neq z$).
- Some voter $k$ can dictate relative rankings of $y$ and $z$ ($\neq x$).

But then, exploiting transitivity, $j$ and $k$ together can dictate relative rankings of $x$ and $z$. As there cannot be two different dictators on the same pair, voters $i$, $j$ and $k$ must all be the same individual. ✓

Thus, voter $i$ is a dictator for any two alternatives, meaning that the rule $F$ is dictatorial. This proves Arrow’s Theorem.
The axiomatic method itself is an (informal) use of logic in SCT.

Next: Several more explicit uses of logic in research on SCT . . .
Formal Modelling of Social Choice Scenarios

Maybe the most obvious application of logic concerns the modelling of social choice scenarios using an appropriate logical language:

- One research direction is to explore how far we can get using a standard logic, such as classical FOL. Do we need second-order constructs to capture IIA? (Grandi and Endriss, 2013)

- Another direction is to design tailor-made logics specifically for SCT (for instance, a modal logic). Can we cast the proof of Arrow’s Theorem in natural deduction? (Ciná and Endriss, 2016)


Automated Reasoning for Social Choice Theory

Logic has long been used to help verify the correctness of hardware and software. Can we use this methodology also here? Yes!

- Automated verification of a (known) proof of Arrow’s Theorem in the HOL proof assistant ISABELLE (Nipkow, 2009).
- Automated proof of Arrow’s Theorem for 3 alternatives and 2 voters using a SAT-solver (Tang and Lin, 2009).
- Automated discovery of new impossibility theorems in ranking sets of objects using a SAT-solver (Geist and Endriss, 2011).


Social Choice in Combinatorial Domains

Suppose 13 voters are asked to each vote Yes or No on three issues; and we use the simple majority rule for each issue independently:

- 3 voters each vote for YNN, NYN, NNY.
- 1 voter each votes for YYY, YYN, YNY, NYY.
- No voter votes for NNN.

But then NNN wins: on each issue, 7 out of 13 vote No (paradox!)

What to do instead? The number of alternatives is exponential in the number of issues (e.g., $2^3 = 8$), so even just representing the voters’ preferences is a challenge (logic-based knowledge representation).


Judgment Aggregation

Preferences are not the only structures you may wish to aggregate . . .

Suppose three robots are in charge of climate control for this building. They need to make judgments on $p$ (the temperature is below $17^\circ C$), on $q$ (we should switch on the heating), and on the “policy” $p \rightarrow q$.

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$p \rightarrow q$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robot 1:</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Robot 2:</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Robot 3:</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

What to do?


We have seen:

- Arrow's Theorem as an example for classical work in social choice theory: there can be no perfect preference aggregation rule.
- Examples for modern research directions in computational social choice (emphasising logic in particular).

For possible entry points into the field, see the papers cited below, my MoL course on COMSOC, or this website:

http://research.illc.uva.nl/COMSOC/
