# SYSU Lectures on the Theory of Aggregation
# Lecture 2: Binary Aggregation with Integrity Constraints

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## Plan for Today

Today's lecture will be devoted to the framework of *binary aggregation with integrity constraints*. Rough outline:

- Old and new examples for aggregation problems and paradoxes
- General perspective on aggregation and paradoxes
- Formal framework of binary aggregation with integrity constraints
- Embedding preference aggregation and judgment aggregation
- New idea: lifting rationality assumptions
- Designing attractive aggregators: representative-voter rules

## Preference Aggregation

**Expert 1:** $\triangle \succ \bigcirc \succ \square$

**Expert 2:** $\bigcirc \succ \square \succ \triangle$

**Expert 3:** $\square \succ \triangle \succ \bigcirc$

**Expert 4:** $\square \succ \triangle \succ \bigcirc$

**Expert 5:** $\bigcirc \succ \square \succ \triangle$

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# Judgment Aggregation

| | $p$ | $p \to q$ | $q$ |
|---|---|---|---|
| **Judge 1:** | True | True | True |
| **Judge 2:** | True | False | False |
| **Judge 3:** | False | True | False |

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## Multiple Referenda

| | *fund museum?* | *fund school?* | *fund metro?* |
|---|---|---|---|
| **Voter 1:** | Yes | Yes | No |
| **Voter 2:** | Yes | No | Yes |
| **Voter 3:** | No | Yes | Yes |

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[ Constraint: we have money for *at most two projects* ]

## General Perspective

The last example is actually pretty general. We can rephrase many aggregation problems as problems of *binary aggregation:*

| | |
|---|---|
| *Do you rank option △ above option ○?* | Yes/No |
| *Do you believe formula "$p \rightarrow q$" is true?* | Yes/No |
| *Do you want the new school to get funded?* | Yes/No |

Each problem domain comes with its own *rationality constraints:*

*Rankings should be transitive and not have any cycles.*

*The accepted set of formulas should be logically consistent.*

*We should fund at most two projects.*

The *paradoxes* we have seen show that the *majority rule* does not *lift* our rationality constraints from the *individual* to the *collective* level.

## Binary Aggregation with Integrity Constraints

Basic terminology and notation:

- Set of *individuals* $\mathcal{N} = \{1, \ldots, n\}$; set of *issues* $\mathcal{I} = \{1, \ldots, m\}$.
- Corresponding set of *propositional symbols* $PS = \{p_1, \ldots, p_m\}$ and *propositional language* $\mathcal{L}_{PS}$ interpreted on $\mathcal{D} = \{0, 1\}^m$.
- An *aggregation rule* is a function $F : \mathcal{D}^n \to \mathcal{D}$. That is, each individual $i \in \mathcal{N}$ votes by submitting a *ballot* $B_i \in \mathcal{D}$.
- An *integrity constraint* is a formula $\mathrm{IC} \in \mathcal{L}_{PS}$ encoding a "rationality assumption". Ballot $B \in \mathcal{D}$ is *rational* iff $B \models \mathrm{IC}$.

U. Grandi and U. Endriss. Binary Aggregation with Integrity Constraints. *Proc. IJCAI-2011*.

# Example

Our multiple-referenda example is formalised as follows:

- Three individuals: $\mathcal{N} = \{1, 2, 3\}$
- Three issues/prop. symbols: $\mathcal{I} = \{\texttt{museum}, \texttt{school}, \texttt{metro}\}$.
- Integrity constraint: $\mathrm{IC} = \neg(\texttt{museum} \land \texttt{school} \land \texttt{metro})$
- Profile: $\boldsymbol{B} = (B_1, B_2, B_3)$ with

$$\begin{aligned} B_1 &= (1, 1, 0) \\ B_2 &= (1, 0, 1) \\ B_3 &= (0, 1, 1) \end{aligned}$$

  Note that $B_i \models \mathrm{IC}$ for all $i \in \{1, 2, 3\}$

- However, $F_{\mathsf{maj}}(\boldsymbol{B}) = (1, 1, 1)$ and $(1, 1, 1) \not\models \mathrm{IC}$.

# Axioms for Binary Aggregation

Classical axioms are easily adapted to this framework. Examples:

- **Unanimity:** For any profile of rational ballots $(B_1, \ldots, B_n)$ and any $x \in \{0, 1\}$, if $b_{i,j} = x$ for all $i \in \mathcal{N}$, then $F(B_1, \ldots, B_n)_j = x$.
- **Anonymity:** For any rational profile $(B_1, \ldots, B_n)$ and any permutation $\pi : \mathcal{N} \to \mathcal{N}$, we get $F(B_1..B_n) = F(B_{\pi(1)}..B_{\pi(n)})$.
- **Independence:** For any issue $j \in \mathcal{I}$ and any two rational profiles $\boldsymbol{B}, \boldsymbol{B'}$, if $b_{i,j} = b'_{i,j}$ for all $i \in \mathcal{N}$, then $F(\boldsymbol{B})_j = F(\boldsymbol{B'})_j$.
- **Issue-Neutrality:** For any two issues $j, j' \in \mathcal{I}$ and any rational profile $\boldsymbol{B}$, if $b_{i,j} = b_{i,j'}$ for all $i \in \mathcal{N}$, then $F(\boldsymbol{B})_j = F(\boldsymbol{B})_{j'}$.
- **Domain-Neutrality:** For any two issues $j, j' \in \mathcal{I}$ and any rational profile $\boldsymbol{B}$, if $b_{i,j} = 1 - b_{i,j'}$ for all $i \in \mathcal{N}$, then $F(\boldsymbol{B})_j = 1 - F(\boldsymbol{B})_{j'}$.

Axioms are (usually) defined for a given *domain of aggregation:* those profiles in $\mathcal{D}^n$ that are rational for a given $\mathrm{IC}$.

# Embedding Preference Aggregation

We can translate Arrovian preference aggregation (for linear orders) into binary aggregation with integrity constraints:

- Introduce propositional symbols $p_{xy}$ to mean "$x$ is better than $y$".
- Include integrity constraints for *irreflexivity* ($\neg p_{xx}$), *completeness* ($p_{xy} \vee p_{yx}$), and *transitivity* ($p_{xy} \wedge p_{yz} \rightarrow p_{xz}$).

Now the *Condorcet paradox* corresponds to this example:

| | $p_{AB}$ | $p_{BC}$ | $p_{AC}$ | *corresponding order* |
|---|---|---|---|---|
| Ann: | 1 | 1 | 1 | $A \succ B \succ C$ |
| Bob: | 0 | 1 | 0 | $B \succ C \succ A$ |
| Cindy: | 1 | 0 | 0 | $C \succ A \succ B$ |

# Embedding Judgment Aggregation

We can also translate formula-based judgment aggregation into binary aggregation with integrity constraints.

- Introduce propositional symbol $p_\varphi$ for every formula $\varphi$ in the agenda $\Phi$.
- Model *completeness* by imposing the IC $p_\varphi \vee p_{\neg\varphi}$ for every non-negated formula $\varphi$ in the agenda $\Phi$.
- Model *consistency* by imposing the IC $\neg(\bigwedge_{\varphi \in S} p_\varphi)$ for every minimally inconsistent subset $S$ of the agenda $\Phi$

Note that from a computational point of view this is not always a good translation (the size of the representation can increase exponentially).

Example: For the agenda $\{p, \neg p, q, \neg q, p \wedge q, \neg(p \wedge q)\}$, we obtain:

$$\begin{aligned}
\mathrm{IC} \quad = \quad & (p_p \vee p_{\neg p}) \wedge (p_q \vee p_{\neg q}) \wedge (p_{p\wedge q} \vee p_{\neg(p\wedge q)}) \wedge \\
& \neg(p_p \wedge p_{\neg p}) \wedge \neg(p_q \wedge p_{\neg q}) \wedge \neg(p_{p\wedge q} \wedge p_{\neg(p\wedge q)}) \wedge \\
& \neg(p_{\neg p} \wedge p_{p\wedge q}) \wedge \neg(p_{\neg q} \wedge p_{p\wedge q}) \wedge \neg(p_p \wedge p_q \wedge p_{\neg(p\wedge q)})
\end{aligned}$$

## Paradoxes

We are now able to give a general definition of "paradox" that captures many of the paradoxes in the literature on social choice theory.

> A *paradox* is a triple $\langle F, \mathrm{IC}, \boldsymbol{B} \rangle$, consisting of an aggregation rule $F$, a profile $\boldsymbol{B}$, and an integritry constraint $\mathrm{IC}$, such that $B_i \models \mathrm{IC}$ for all individuals $i \in \mathcal{N}$ but $F(\boldsymbol{B}) \not\models \mathrm{IC}$.

## Collective Rationality

An aggregation rule $F : \mathcal{D}^n \to \mathcal{D}$ is *collectively rational* for $\mathrm{IC} \in \mathcal{L}_{PS}$ if $B_i \models \mathrm{IC}$ for all $i \in \mathcal{N}$ implies $F(B_1, \ldots, B_n) \models \mathrm{IC}$.

That is, $F$ is collectively rational for $\mathrm{IC}$, if there exists not profile $\boldsymbol{B}$ such that $\langle F, \mathrm{IC}, \boldsymbol{B} \rangle$ is a paradox.

We also say: $F$ can *lift* $\mathrm{IC}$ from the individual to the collective level.

# Template for Results

Let $\mathcal{L} \subseteq \mathcal{L}_{PS}$ be a *language of integrity constraints*. By fixing $\mathcal{L}$ we fix a range of possible domains of aggregation (one for each $\mathrm{IC} \in \mathcal{L}$).

Two ways of defining classes of aggregation rules:

- The class of rules defined by a given list of *axioms* AX:

$$\mathcal{F}_{\mathcal{L}}[\text{AX}] \quad := \quad \{F : \mathcal{D}^n \to \mathcal{D} \mid F \text{ satisfies AX on all } \mathcal{L}\text{-domains}\}$$

- The class of rules that *lift* all integrity constraints in $\mathcal{L}$:

$$\mathcal{CR}[\mathcal{L}] \quad := \quad \{F : \mathcal{D}^n \to \mathcal{D} \mid F \text{ is collect. rat. for all } \mathrm{IC} \in \mathcal{L}\}$$

**What we want:**

$$\mathcal{CR}[\mathcal{L}] \;=\; \mathcal{F}_{\mathcal{L}}[\text{AX}]$$

# Example for a Characterisation Result

**Theorem 1** *$F$ will lift all integrity constraints that can be expressed as a conjunction of literals ("cube") if and only if $F$ is unanimous:*

$$\mathcal{CR}[\text{cubes}] \quad = \quad \mathcal{F}_{\text{cubes}}[\text{Unanimity}]$$

U. Grandi and U. Endriss. Lifting Integrity Constraints in Binary Aggregation. *Artificial Intelligence*, 199–200:45–66, 2013.

# More Results

**Characterisation results (selection):**

- $F$ lifts all constraints $p_j \leftrightarrow p_k$ *iff* $F$ is *issue-neutral*
- $F$ lifts all constraints $p_j \leftrightarrow \neg p_k$ *iff* $F$ is *domain-neutral*

**Negative results:**

- there exists *no language* that characterises *anonymous* rules
- there exists *no language* that characterises *independent* rules

U. Grandi and U. Endriss. Lifting Integrity Constraints in Binary Aggregation. *Artificial Intelligence*, 199–200:45–66, 2013.

# Example

Suppose 13 voters are asked to each vote *yes* or *no* on three issues; and we use the majority rule for each issue independently:

- 3 voters each vote for YNN, NYN, NNY.
- 1 voter each votes for YYY, YYN, YNY, NYY.
- No voter votes for NNN.

NNN wins: 7 out of 13 vote *no* on each issue.

This is an instance of the *paradox of multiple elections:* the winning combination received the fewest number of (actually: *no*) votes.

▶ But is it a paradox according to our definition?

S.J. Brams, D.M. Kilgour, and W.S. Zwicker. The Paradox of Multiple Elections. *Social Choice and Welfare*, 15(2):211–236, 1998.

## Designing Good Aggregation Rules

We want to identify good methods for binary aggregating.

- Problem: the simple methods people usually use ("issue-wise majority") can lead to paradoxical outcomes.
- Problem: more sophisticated methods ("distance-based") are computationally intractable (as we will see).
- New idea: use an aggregation rule that identifies the "most representative" voter and just copies that voter's ballot.

Take-home message will be: simple, but works surprisingly well.

U. Endriss and U. Grandi. Binary Aggregation by Selection of the Most Representative Voter. *Proc. MPREF-2013*.

## Distance-based Aggregation

How to avoid paradoxes?

→ Only consider outcomes that respect the integrity constraint.
→ Which one to pick?—the one "closest" to the individual inputs.

These considerations suggest the following rule:

- The (Hamming) *distance* between an individual input and the outcome is the number of "point decisions" on which they differ.
- Elect the (consistent/rational) outcome that *minimises* the sum of distances to the individual inputs! (+ break ties if needed)

For preference aggregation (with "point decisions" being pairwise rankings), this is the famous *Kemeny rule*. No rule is perfect, but many consider this one to be pretty much the best there is.

But: this is $\Theta_2^p$-*complete* ("complete for parallel access to NP"). ☹

## Taming the Complexity

Where does this complexity come from?
$\longrightarrow$ We need to search through all candidate outcomes.

- there might be exponentially many of those
- for each of them, checking consistency might be nontrivial

An idea:

- restrict set of choices to a small set of candidate outcomes
- make sure you can be certain all candidate outcomes are consistent

The easiest way of doing this:

candidate outcomes = choices made by individuals ("support")

# Example

Find the outcome that minimises the sum of distances for this profile:

| **Issue:** | **1** | **2** | **3** |
|---|---|---|---|
| 20 voters: | 0 | 1 | 1 |
| 10 voters: | 1 | 0 | 1 |
| 11 voters: | 1 | 1 | 0 |

Solution: $(1, 1, 1)$. The distance is 41 (41 voters $\times$ 1 disagreement).

Note: same as majority outcome (as there's no integrity constraint).

Now suppose there's an IC that says that $(1, 1, 1)$ is not ok.

## Example (continued)

Find the outcome that minimises the sum of distances for this profile:

| **Issue:** | **1** | **2** | **3** |
|---|---|---|---|
| 20 voters: | 0 | 1 | 1 |
| 10 voters: | 1 | 0 | 1 |
| 11 voters: | 1 | 1 | 0 |

"Average voter" says: $(0, 1, 1)$.
The distance is 42 (20 with no disagreements $+$ 21 with 2 each).

So: not much worse (42 *vs.* 41), but easier to find (choose from 3 rather than $2^3 = 8$ outcomes; all 3 known to be consistent *a priori*)

## Additional Notation and Terminology

- *Hamming distance* between ballots: $H(B, B') = |\{j \in \mathcal{I} \mid b_j \neq b'_j\}|$ and between a ballot and a profile: $\mathcal{H}(B, \boldsymbol{B}) = \sum_{i \in \mathcal{N}} H(B, B_i)$.
- *Support* of profile $\boldsymbol{B}$: $\textsc{Supp}(\boldsymbol{B}) = \{B_1\} \cup \cdots \cup \{B_n\}$.

# Rules Based on Representative Voters

Idea: Choose an outcome by first choosing a voter (based on the input profile) and then copying that voter's ballot.

$$\text{Fix } g : \mathcal{D}^n \to \mathcal{N}. \text{ Then let } F : \boldsymbol{B} \mapsto B_{g(\boldsymbol{B})}.$$

Good properties (of all these rules):

- *No paradoxes* ever, whatever the IC (not true for any other rule).
- *Unanimity* guaranteed. [obvious]
- *Neutrality* (both kinds) guaranteed. [maybe less obvious]
- *Low complexity* for natural choices of $g$.

But:

- Includes some really bad rules, such as Arrovian *dictatorships:*

$$g \equiv i, \text{ i.e., } F : (B_1, \ldots, B_n) \mapsto B_i \text{ with } i \text{ being the dictator}$$

U. Endriss and U. Grandi. Binary Aggregation by Selection of the Most Representative Voter. *Proc. MPREF-2013.*

## Two Representative-Voter Rules

The *average-voter rule* selects those individual ballots that minimise the Hamming distance to the profile:

$$\mathrm{AVR}(\boldsymbol{B}) \quad = \quad \operatorname*{argmin}_{B \in \textsc{Supp}(\boldsymbol{B})} \; \mathcal{H}(B, \boldsymbol{B})$$

Remark: if you replace the set $\textsc{Supp}(\boldsymbol{B})$ by $\mathrm{Mod}(\mathrm{IC})$, the set of *all* consistent outcomes, you obtain the full distance-based rule.

The *majority-voter rule* selects those individual ballots that minimise the Hamming distance to one of the majority outcomes:

$$\mathrm{MVR}(\boldsymbol{B}) \quad = \quad \operatorname*{argmin}_{B \in \textsc{Supp}(\boldsymbol{B})} \; \min\{H(B, B') \mid B' \in \mathrm{Maj}(\boldsymbol{B})\}$$

Connections:

- $\mathrm{AVR}$ related to *Kemeny* rule in voting/preference aggregation.
- $\mathrm{MVR}$ related to *Slater* rule in voting/preference aggregation.

# Example

The AVR and the MVR really can give different outcomes:

| **Issue:** | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| 1 voter: | 0 | 1 | 1 | 1 | 1 |
| 2 voters: | 1 | 0 | 0 | 0 | 0 |
| 10 voters: | 0 | 1 | 1 | 0 | 0 |
| 10 voters: | 0 | 0 | 0 | 1 | 1 |
| Maj: | 0 | 0 | 0 | 0 | 0 |
| MVR: | 1 | 0 | 0 | 0 | 0 |
| AVR: | 0 | 0 | 0 | 1 | 1 |

Remark: This is the AVR-winner for one way of breaking ties (for the other way it is also different from the MVR-winner).

## Which rule is better?

We will compare the AVR and the MVR according to

- algorithmic efficiency [MVR wins]
- satisfaction of a choice-theoretic axiom [AVR wins]
- relative distance to the input profile [AVR wins]

# Algorithmic Efficiency

Recall: $m$ is the number of issues; $n$ is the number of voters.

Winner determination for the $\mathrm{MVR}$ is in $O(mn)$:

- compute the majority vector in $O(mn)$
- compare each ballot to the majority vector in $O(mn)$

Winner determination for the $\mathrm{AVR}$ is in $O(mn \log n)$:

- compute the vector of sums in $O(mn)$
- compute the difference between each ballot (multiplied by $n$) to the vector of sums in $O(mn \log n)$
  [$O(\log n)$ because we are working with integers up to $n$]

So: both rules are efficient, but the $\mathrm{MVR}$ more so.

# Axiom: Reinforcement

We are looking for an axiom that separates the two rules ...

$F$ satisfies *reinforcement* if for any two profiles $\boldsymbol{B}$ and $\boldsymbol{B}'$ with

- $\textsc{Supp}(\boldsymbol{B}) = \textsc{Supp}(\boldsymbol{B}')$ and
- $F(\boldsymbol{B}) \cap F(\boldsymbol{B}') \neq \emptyset$

it is the case that $F(\boldsymbol{B} \oplus \boldsymbol{B}') = F(\boldsymbol{B}) \cap F(\boldsymbol{B}')$.

This is a natural requirement: if two groups independently agree that a certain outcome is best, we would expect them to uphold this choice when choosing together.

**Theorem 2** *The* $\mathrm{AVR}$ *satisfies reinforcement, but the* $\mathrm{MVR}$ *does not.*

U. Endriss and U. Grandi. Binary Aggregation by Selection of the Most Representative Voter. *Proc. MPREF-2013*.

## Relative Distance to the Input Profile

Both rules select from $\textsc{Supp}(\boldsymbol{B})$ and the $\mathrm{AVR}$ by definition picks the candidate outcome closest to the profile. Thus:

> Fact: The Hamming distance between the (worst) $\mathrm{AVR}$-winner and the profile never exceeds the Hamming distance between the (best) $\mathrm{MVR}$-winner and the profile.

More importantly, as we shall see next, both rules are very good approximations of the full distance-based rule ...

# Approximation Results

$F$ is said to be an *$\alpha$-approximation* of $F'$ if for every profile $\boldsymbol{B}$:

$$\max \mathcal{H}(F(\boldsymbol{B}), \boldsymbol{B}) \leqslant \alpha \cdot \min \mathcal{H}(F'(\boldsymbol{B}), \boldsymbol{B})$$

If $F'$ is a "nice" but computationally intractable rule and if $\alpha$ is a constant, then this would be considered great news for $F$.

**Theorem 3** *The* $\mathrm{AVR}$ *and the* $\mathrm{MVR}$ *are (strict)* $2$*-approximations of the full distance-based rule (for any IC).*

Proof: next slide

An important additional insight here is that approximations get better as we increase the logical strength of the IC (reason: the stronger $\mathrm{IC}$, the fewer outcome the distance-based rule can choose from).

U. Endriss and U. Grandi. Binary Aggregation by Selection of the Most Representative Voter. *Proc. MPREF-2013*.

# Proof Sketch

We will prove that the $\mathrm{MVR}$ is a (strict) 2-approx. of the *majority rule* ($=$ distance-based rule with $\mathrm{IC} = \top$). All other claims then follow.

To simplify presentation, suppose there is only a single majority winner. W.l.o.g., suppose it is $(0, \ldots, 0)$.

Let $m_i$ be the number of issues labelled as 1 by individual $i$. Let $i^\star$ be the voter selected by the $\mathrm{MVR}$, i.e., $m_{i^\star} \leqslant m_i$ for all $i \in \mathcal{N}$.

If $m_{i^\star} = 0$, then we are done (approx. ratio 1). So suppose $m_{i^\star} \neq 0$.

We need to show:

$$\sum_{i \in \mathcal{N}} H(B_{i^\star}, B_i) \quad < \quad 2 \cdot \sum_{i \in \mathcal{N}} m_i$$

But this is the case:

- $H(B_{i^\star}, B_i) \leqslant m_{i^\star} + m_i \leqslant 2 \cdot m_i$ for all $i \neq i^\star$ (triangle inequality)
- $H(B_{i^\star}, B_{i^\star}) = 0 < 2 \cdot m_{i^\star}$ ✓

## Summary

Binary aggregation with integrity constraints:

- *language* to express *rationality assumptions* in binary aggregation
- concept of *collective rationality* with respect to an IC
- characterisation results, relating *axioms* and *languages*
- application: *embedding* preference + judgment aggregation
- application: design of aggregation rules that avoid all paradoxes (*representative-voter rules* have surprisingly good properties)

In principle, *any* aggregation problem can be modelled using binary aggregation. But sometimes a more domain-specific framework will be more insightful and/or will have better algorithmic properties.

For an introduction to binary aggregation with integrity constraints, consult the paper cited below.

U. Grandi and U. Endriss. Binary Aggregation with Integrity Constraints. *Proc. IJCAI-2011*.