SYSU Lectures on the Theory of Aggregation
Lecture 3: Graph Aggregation

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Plan for Today

This will be a lecture on the relatively new framework of graph aggregation, which in terms of level of abstraction is located somewhere between preference and binary aggregation.

- Formal framework and axioms
- Example for a characterisation result: quota rules
- Collective rationality and some simple possibility results
- A general impossibility result (generalising Arrow’s Theorem)
- Using modal logic to specify collective rationality requirements
**Graph Aggregation**

Fix a finite set of *vertices* $V$. A (directed) *graph* $G = \langle V, E \rangle$ over $V$ is defined by a set of *edges* $E \subseteq V \times V$ [so we can talk about $E$, not $G$].

Each member of a finite set of *individuals* $\mathcal{N} = \{1, \ldots, n\}$ provides such a graph, giving rise to a *profile* $E = (E_1, \ldots, E_n)$.

An *aggregation rule* is a function mapping profiles to collective graphs:

$$F : (2^{V \times V})^n \rightarrow 2^{V \times V}$$

Example: *majority rule* (accept an edge *iff* $> \frac{n}{2}$ of the individuals do)

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Special Case: Preference Aggregation

Preference aggregation, with preferences being strict *linear orders* (as discussed on Tuesday), is a special case of graph aggregation:

- vertices = alternatives
- edges = *preferred-to* relation

Preference aggregation, with preferences being *weak orders* (another standard framework) is also a special case of graph aggregation.

On the other hand, graph aggregation is itself a special case of the framework of *binary aggregation* (issues = edges).
Applications

Graphs are everywhere. Examples for recent work that can be cast as instances of graph aggregation:

- Nonstandard preference aggregation, e.g., when preferences are taken to be partial orders to account for bounded rationality (Pini et al., 2009).
- Merging argumentation frameworks (e.g., Coste-Marquis et al., 2007).
- Aggregation of different logics, with edges corresponding to consequence relations (Wen and Liu, 2013).

Another promising area might be the merging of social networks.


Axioms

We may want to impose certain axioms on $F : (2^{V \times V})^n \to 2^{V \times V}$, e.g.:

- **Anonymous**: $F(E_1, \ldots, E_n) = F(E_{\pi(1)}, \ldots, E_{\pi(n)})$
- **Nondictatorial**: for no $i^* \in N$ you always get $F(E) = E_{i^*}$
- **Unanimous**: $F(E) \supseteq E_1 \cap \cdots \cap E_n$
- **Grounded**: $F(E) \subseteq E_1 \cup \cdots \cup E_n$
- **Neutral**: $N_e^E = N_{e'}^E$ implies $e \in F(E) \iff e' \in F(E)$
- **Independent**: $N_e^E = N_{e'}^E$ implies $e \in F(E) \iff e \in F(E')$
- **Monotonic**: $e \in F(E)$ implies $e \in F(E')$ whenever $E'$ is obtained from $E$ by having one additional individual accept $e$

For technical reasons, we’ll restrict some axioms to nonreflexive edges $(x, y) \in V \times V$ with $x \neq y$ (NR-neutral, NR-nondictatorial).

**Notation**: $N_e^E = \{ i \in N \mid e \in E_i \} = \text{coalition accepting edge } e \text{ in } E$
Quota Rules

A quota rule is an aggregation rule $F_q$, defined via a function $q : V \times V \to \{0, 1, \ldots, n, n+1\}$, such that for every profile $E$:

$$F_q(E) = \{ e \in V \times V \mid |N_e^E| \geq q(e) \}$$

$F_q$ is called uniform if $q$ is a constant function.

Examples:

- Strict majority rule: $q \equiv \lceil \frac{n+1}{2} \rceil$
- Union rule: $q \equiv 1$, i.e., $F_q(E) = E_1 \cup \cdots \cup E_n$
- Intersection rule: $q \equiv n$, i.e., $F_q(E) = E_1 \cap \cdots \cap E_n$
- Trivial quota rules (constant): $q \equiv 0$ or $q \equiv n + 1$
Characterisation

Adapting similar results in judgment aggregation due to Dietrich and List (2007), we obtain the following characterisation:

**Proposition 1** An aggregator is anonymous, independent, and monotonic if and only if it is a quota rule.

Proof sketch: \((\Leftarrow)\) Clear. ✓

\((\Rightarrow)\) By independence, decision on \(e\) only depends on \(N_e^E\). By anonymity, only \(|N_e^E|\) matters. By monotonicity, “no gaps”. ✓

Furthermore:

- Adding *neutrality*, we get uniform quota rules.
- Adding *unanimity* and *groundedness*, we get nontrivial rules.

Collective Rationality

Aggregator $F$ is collectively rational (CR) for graph property $P$ if, whenever all individual graphs $E_i$ satisfy $P$, so does the outcome $F(E)$.

Examples for graph properties: reflexivity, transitivity, seriality, . . .
Example

Three agents each provide a graph on the same set of four vertices:

If we aggregate using the *majority rule*, we obtain this graph:

Observations:
- Majority rule not collectively rational for *seriality*.
- But *symmetry* is preserved.
- So is *reflexivity* (easy: individuals violate it).
Two Simple Possibility Results

The fact that the example worked for reflexivity is no coincidence:

**Proposition 2** Any *unanimous* aggregator is CR for *reflexivity*.

**Proof:** If every individual graph includes edge \((x, x)\), then unanimity ensures the same for the collective outcome graph. ✓

By a similar argument, we obtain:

**Proposition 3** Any *grounded* aggregator is CR for *irreflexivity*.
Recall: Arrow’s Theorem

This is how we had phrased Arrow’s Theorem on Tuesday:

**Theorem 4 (Arrow, 1951)** Any SWF for \( \geq 3 \) alternatives that satisfies the Pareto condition and IIA must be a dictatorship.

This is the version for strict linear orders (Arrow’s original formulation was for weak orders, which doesn’t make much of a difference though).

I still owe you a proof.

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**Arrow’s Theorem in Graph Aggregation**

Our formulation in graph aggregation:

For $|V| \geq 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for irreflexivity, transitivity, and completeness.

This implies the standard formulation, because:

- preferences (linear orders) = irreflexive, transitive, complete graphs
- nondictatorial = NR-nondictatorial for irreflexive graphs
- (weak) Pareto $\Rightarrow$ unanimous $+$ grounded
- CR for irreflexivity is vacuous (implied by groundedness)

Main question for the next part:

- For what other classes of graphs does this go through?
Winning Coalitions

If an aggregator $F$ is *independent*, then for every edge $e$ there exists a set of *winning coalitions* $\mathcal{W}_e \subseteq 2^\mathcal{N}$ such that $e \in F(E) \iff N^E_e \in \mathcal{W}_e$.

Furthermore:

- If $F$ is *unanimous*, then $\mathcal{N} \in \mathcal{W}_e$ for all edges $e$.
- If $F$ is *grounded*, then $\emptyset \notin \mathcal{W}_e$ for all edges $e$.
- If $F$ is *neutral*, then there is one $\mathcal{W}$ with $\mathcal{W} = \mathcal{W}_e$ for all edges $e$. 
Proof Plan

Given: Arrovian aggregator $F$ (*unanimous*, *grounded*, *independent*)

Want: Impossibility for *collective rationality* for graph property $P$

This will work if $P$ is *contagious*, *implicative*, and *disjunctive* (TBD).

Lemma: CR for *contagious* $P$ $\Rightarrow$ $F$ is NR-neutral.

$\Rightarrow$ $F$ characterised by some $\mathcal{W}$: $(x, y) \in F(E) \iff N^E_{(x,y)} \in \mathcal{W}$ [$x \neq y$]

Lemma: CR for *implicative* & *disjunctive* $P$ $\Rightarrow$ $\mathcal{W}$ is an *ultrafilter*, i.e.:

(i) $\emptyset \notin \mathcal{W}$ [this is immediate from groundedness]

(ii) $C_1, C_2 \in \mathcal{W}$ implies $C_1 \cap C_2 \in \mathcal{W}$ (closure under intersections)

(iii) $C$ or $\mathcal{N} \setminus C$ is in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$ (maximality)

$\mathcal{N}$ is *finite* $\Rightarrow$ $\mathcal{W}$ is *principal*: $\exists i^* \in \mathcal{N}$ s.t. $\mathcal{W} = \{C \in 2^\mathcal{N} | i^* \in C\}$

But this just means that $i^*$ is a dictator: $F$ is (NR-) *dictatorial*. ✓
Neutrality Lemma

Consider any Arrovian aggregator (unanimous, grounded, independent).

Call a property $P$ \textit{xy/zw-contagious} if there exist disjoint $S^+, S^- \subseteq V \times V$ s.t. every graph $E \in P$ satisfies $[\land S^+ \land \neg \lor S^-] \rightarrow [xEy \rightarrow zEw]$.

\textit{CR for xy/zw-contagious} $P$ implies: coalition $C \in \mathcal{W}_{(x,y)} \Rightarrow C \in \mathcal{W}_{(z,w)}$

Call $P$ \textit{contagious} if it satisfies (at least) one of the three conditions below:

(i) $P$ is $xy/yz$-contagious for all $x, y, z \in V$.
(ii) $P$ is $xy/zx$-contagious for all $x, y, z \in V$.
(iii) $P$ is $xy/xz$-contagious and $xy/zy$-contagious for all $x, y, z \in V$.

Example: Transitivity ($[yEz] \rightarrow [xEy \rightarrow xEz]$ and $[zEx] \rightarrow [xEy \rightarrow zEy]$)

Contagiousness allows us to reach every NR edge from every other NR edge. Thus, \textit{CR for contagious} $P$ implies $\mathcal{W}_e = \mathcal{W}_{e'}$ for all NR edges $e, e'$.

So: \textit{Collective rationality} for a \textit{contagious} property implies NR-neutrality.
Ultrafilter Lemma

Let $F$ be unanimous, grounded, independent, NR-neutral, and CR for $P$. So there exists a family of winning coalitions $\mathcal{W}$ s.t. $e \in F(E) \Leftrightarrow N_e^E \in \mathcal{W}$.

Show that $\mathcal{W}$ is an ultrafilter (under certain assumptions on $P$):

(i) $\emptyset \notin \mathcal{W}$: immediate form groundedness

(ii) Closure under intersections: $C_1, C_2 \in \mathcal{W} \Rightarrow C_1 \cap C_2 \in \mathcal{W}$

Call $P$ implicative if there exist disjoint sets $S^+, S^- \subseteq V \times V$ and distinct edges $e_1, e_2, e_3 \in V \times V \setminus (S^+ \cup S^-)$ s.t. all graphs $E \in P$ satisfy $[\bigwedge S^+ \land \neg \bigvee S^-] \rightarrow [e_1 \land e_2 \rightarrow e_3]$. Example: transitivity

CR for implicative $P \Rightarrow$ closure under intersections

Proof: Consider a profile where $C_1$ accept $e_1$, $C_2$ accept $e_2$, $C_1 \cap C_2$ accept $e_3$, everyone accepts $S^+$, and nobody accepts any edge in $S^-$. ✓
Ultrafilter Lemma (continued)

Still showing that $\mathcal{W}$ is an *ultrafilter* (for certain assumptions on $P$):

(iii) **Maximality**: $C$ or $\mathcal{N} \setminus C$ in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$

Call $P$ *disjunctive* if there exist disjoint sets $S^+, S^- \subseteq V \times V$ and distinct edges $e_1, e_2 \in V \times V \setminus (S^+ \cup S^-)$ s.t. all graphs $E \in P$ satisfy $[\land S^+ \land \neg \lor S^-] \rightarrow [e_1 \lor e_2]$.

Example: completeness

CR for disjunctive $P \Rightarrow$ maximality

**Proof**: Consider a profile where $C$ accept $e_1$, $\mathcal{N} \setminus C$ accept $e_2$, everyone accepts $S^+$, and nobody accepts any edge in $S^-$. ✓
End of Proof: Dictatorship

We have shown that our assumptions imply that $F$ is characterised by a \textit{single family} $\mathcal{W}$ of winning coalitions ($(x, y) \in F(E) \iff N^E_{(x,y)} \in \mathcal{W}$ for $x \neq y$) and that $\mathcal{W}$ must be an \textit{ultrafilter}:

(i) $\emptyset \notin \mathcal{W}$
(ii) $C_1, C_2 \in \mathcal{W}$ implies $C_1 \cap C_2 \in \mathcal{W}$ (closure under intersections)
(iii) $C$ or $\mathcal{N} \setminus C$ is in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$ (maximality)

Take the \textit{intersection} of all winning coalitions (possible, as $\mathcal{N}$ is \textit{finite}).

By (ii), this must be a winning coalition itself.
By (i), not empty. By (iii) cannot have two or more elements.
Thus, it must be a singleton $\{i^*\}$, meaning that $i^*$ is a dictator. $\checkmark$
**General Impossibility Theorem**

We have seen a proof for the following theorem:

**Theorem 5** *For* $|V| \geq 3$, *there exists no* NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for any graph property that is contagious, implicative, and disjunctive.

Many combinations of graph properties have our meta-properties:

<table>
<thead>
<tr>
<th>Property</th>
<th>C/I/D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Transitivity</strong></td>
<td>$\forall xyz. (xEy \land yEz \rightarrow xEz)$</td>
</tr>
<tr>
<td>Right Euclidean</td>
<td>$\forall xyz. (xEy \land xEz \rightarrow yEz)$</td>
</tr>
<tr>
<td>Left Euclidean</td>
<td>$\forall xyz. (xEy \land zEy \rightarrow zEx)$</td>
</tr>
<tr>
<td>Seriality</td>
<td>$\forall x. \exists y. xEy$</td>
</tr>
<tr>
<td>Completeness</td>
<td>$\forall xy. [x \neq y \rightarrow (xEy \lor yEx)]$</td>
</tr>
<tr>
<td>Connectedness</td>
<td>$\forall xyz. (xEy \land xEz \rightarrow (yEz \lor zEy))$</td>
</tr>
<tr>
<td>Negative Transitivity</td>
<td>$\forall xyz. [xEy \rightarrow (xEz \lor zEy)]$</td>
</tr>
</tbody>
</table>

**Arrow’s Theorem**: use transitivity and completeness $\checkmark$
Collective Rationality and Modal Logic

Modal logic is a useful language for talking about graphs. This suggests trying to express CR requirements in modal logic. On the following slides, we will see some preliminary results in this directions:

- The modal logic perspective suggests a differentiation into three levels of collective rationality.

- For properties expressible as modal logic formulas satisfying certain syntactic constraints, we obtain simple possibility results.

I shall assume familiarity with basic modal logic.
Levels of Collective Rationality

Graphs $\langle V, E \rangle$ may be considered Kripke frames. The semantics of modal logic suggests three levels of collective rationality:

- $F$ is \textit{frame-CR} for a modal integrity constraint $\varphi$ if $\langle V, E_i \rangle \models \varphi$ for all $i \in \mathcal{N}$ implies $\langle V, F(E) \rangle \models \varphi$.

- $F$ is \textit{model-CR} for a modal IC $\varphi$ if for all valuations $Val: \Phi \to 2^V$ $\langle \langle V, E_i \rangle, Val \rangle \models \varphi$ for all $i \in \mathcal{N}$ implies $\langle \langle V, F(E) \rangle, Val \rangle \models \varphi$.

- $F$ is \textit{world-CR} for a modal IC $\varphi$ if for all valuations $Val: \Phi \to 2^V$ and worlds $x \in V$ we have $\langle \langle V, E_i \rangle, Val \rangle, x \models \varphi$ for all $i \in \mathcal{N}$ implying $\langle \langle V, F(E) \rangle, Val \rangle, x \models \varphi$.

Via modal correspondence theory, frame-CR corresponds to our original notion of collective rationality.
Connections

**Proposition 6** Let $F$ be an aggregator and let $\varphi$ a modal integrity constraint. Then the following implications hold:

(i) If $F$ is world-CR for $\varphi$, then $F$ is also model-CR for $\varphi$.

(ii) If $F$ is model-CR for $\varphi$, then $F$ is also frame-CR for $\varphi$.

These implications are strict. Example:

Suppose $F$ returns the full graph if all individual graphs satisfy $\Diamond(p \lor \neg p)$, and the empty graph otherwise. Then $F$ is model-CR but not world-CR for $\Diamond(p \lor \neg p)$: Take a profile of graphs with two worlds where $E_i = \{(x, y)\}$ for all $i \in \mathcal{N}$. The outcome returned by $F$ is the empty graph, in violation of world-CR for $\Diamond(p \lor \neg p)$ at world $x$.

**Remark:** Impossibility results are most interesting for frame-CR. Possibility results are most interesting for world-CR.
Possibility Results

Let us call a $\square$-formula any formula in NNF without any occurrences of $\Diamond$ (and define $\Diamond$-formulas accordingly).

**Proposition 7** If an aggregator $F$ is such that for every profile $E$ there exists an individual $i^* \in \mathbb{N}$ such that $F(E) \subseteq E_{i^*}$, then $F$ is world-CR for all $\square$-formulas.

**Proposition 8** If an aggregator $F$ is such that for every profile $E$ there exists an individual $i^* \in \mathbb{N}$ such that $F(E) \supseteq E_{i^*}$, then $F$ is world-CR for all $\Diamond$-formulas.

**Proposition 9** If an aggregator $F$ is such that for every profile $E$ there exists an individual $i^* \in \mathbb{N}$ such that $F(E) = E_{i^*}$, then $F$ is world-CR for all modal integrity constraints.

This last result is related to the fact that no representative-voter rule can ever cause a paradox (lecture on binary aggregation).
Summary

We have introduced graph aggregation as a generalisation of preference aggregation and then considered collective rationality.

Why is this interesting?

• Potential for applications: abstract argumentation, social networks
• Deep insights into the structure of impossibilities: direct link between CR requirements and neutrality ultrafilter conditions

Topics covered:

• Axiomatic characterisation of quota rules
• Simple possibility results (e.g., unanimity lifting reflexivity)
• General impossibility theorem, ultrafilter proof technique
• The modal logic perspective