SYSU Lectures on the Theory of Aggregation
Lecture 3: Graph Aggregation

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Plan for Today

This will be a lecture on the relatively new framework of graph aggregation, which in terms of level of abstraction is located somewhere between preference and binary aggregation.

- Formal framework and axioms
- Example for a characterisation result: quota rules
- Collective rationality and some simple possibility results
- A general impossibility result (generalising Arrow's Theorem)
- Using modal logic to specify collective rationality requirements

Graph Aggregation

Fix a finite set of *vertices* V. A (directed) *graph* $G = \langle V, E \rangle$ over V is defined by a set of *edges* $E \subseteq V \times V$ [so we can talk about E, not G].

Each member of a finite set of *individuals* $\mathcal{N} = \{1, \ldots, n\}$ provides such a graph, giving rise to a *profile* $\boldsymbol{E} = (E_1, \ldots, E_n)$.

An *aggregation rule* is a function mapping profiles to collective graphs:

$$F : (2^{V \times V})^n \to 2^{V \times V}$$

Example: *majority rule* (accept an edge *iff* $> \frac{n}{2}$ of the individuals do)

U. Endriss and U. Grandi. Graph Aggregation. *Proc. COMSOC-2012.*

Special Case: Preference Aggregation

Preference aggregation, with preferences being strict *linear orders* (as discussed on Tuesday), is a special case of graph aggregation:

- vertices = alternatives
- edges = *preferred-to* relation

Preference aggregation, with preferences being *weak orders* (another standard framework) is also a special case of graph aggregation.

On the other hand, graph aggregation is itself a special case of the framework of *binary aggregation* (issues = edges).

Applications

Graphs are everywhere. Examples for recent work that can be cast as instances of graph aggregation:

- Nonstandard preference aggregation, e.g., when preferences are taken to be *partial orders* to account for bounded rationality (Pini et al., 2009).
- Merging *argumentation frameworks* (e.g., Coste-Marquis et al., 2007).
- Aggregation of different *logics*, with edges corresponding to consequence relations (Wen and Liu, 2013).

Another promising area might be the merging of *social networks*.

M.S. Pini, F. Rossi, K.B. Venable, and T. Walsh, Aggregating Partially Ordered Preferences. *Journal of Logic and Computation*, 19(3):475–502, 2009.

S. Coste-Marquis, C. Devred, S. Konieczny, M.-C. Lagasquie-Schiex, and P. Marquis. On the Merging of Dung's Argumentation Systems. *Artificial Intelligence*, 171(10–15):730–753, 2007.

X. Wen and H. Liu. Logic Aggregation. *Proc. LORI-2013*.

Axioms

We may want to impose certain *axioms* on $F : (2^{V \times V})^n \to 2^{V \times V}$, e.g.:

- *Anonymous:* $F(E_1, \ldots, E_n) = F(E_{\pi(1)}, \ldots, E_{\pi(n)})$
- *Nondictatorial:* for no $i^\star \in \mathcal{N}$ you always get $F(\boldsymbol{E}) = E_{i^\star}$
- *Unanimous:* $F(\boldsymbol{E}) \supseteq E_1 \cap \cdots \cap E_n$
- *Grounded:* $F(\boldsymbol{E}) \subseteq E_1 \cup \cdots \cup E_n$
- *Neutral:* $N_e^{\boldsymbol{E}} = N_{e'}^{\boldsymbol{E}}$ implies $e \in F(\boldsymbol{E}) \Leftrightarrow e' \in F(\boldsymbol{E})$
- *Independent:* $N_e^{\boldsymbol{E}} = N_e^{\boldsymbol{E'}}$ implies $e \in F(\boldsymbol{E}) \Leftrightarrow e \in F(\boldsymbol{E'})$
- *Monotonic:* $e \in F(\boldsymbol{E})$ implies $e \in F(\boldsymbol{E'})$ whenever $\boldsymbol{E'}$ is obtained from $\boldsymbol{E}$ by having one additional individual accept e

For technical reasons, we'll restrict some axioms to *nonreflexive edges* $(x, y) \in V \times V$ with $x \neq y$ (NR-neutral, NR-nondictatorial).

<u>Notation:</u> $N_e^{\boldsymbol{E}} = \{i \in \mathcal{N} \mid e \in E_i\}$ = *coalition* accepting edge e in $\boldsymbol{E}$

Quota Rules

A *quota rule* is an aggregation rule F_q, defined via a function $q : V \times V \to \{0, 1, \ldots, n, n{+}1\}$, such that for every profile $\boldsymbol{E}$:

$$F_q(\boldsymbol{E}) \quad = \quad \{e \in V \times V \mid |N_e^{\boldsymbol{E}}| \geqslant q(e)\}$$

F_q is called *uniform* if q is a constant function.

Examples:

- Strict majority rule: $q \equiv \lceil \frac{n+1}{2} \rceil$
- Union rule: $q \equiv 1$, i.e., $F_q(\boldsymbol{E}) = E_1 \cup \cdots \cup E_n$
- Intersection rule: $q \equiv n$, i.e., $F_q(\boldsymbol{E}) = E_1 \cap \cdots \cap E_n$
- Trivial quota rules (constant): $q \equiv 0$ or $q \equiv n + 1$

Characterisation

Adapting similar results in judgment aggregation due to Dietrich and List (2007), we obtain the following characterisation:

Proposition 1 *An aggregator is anonymous, independent, and monotonic if and only if ist is a quota rule.*

Proof sketch: $(\Leftarrow)$ Clear. ✓
$(\Rightarrow)$ By independence, decision on e only depends on $N_e^{\boldsymbol{E}}$. By anonymity, only $|N_e^{\boldsymbol{E}}|$ matters. By monotonicity, "no gaps". ✓

Furthermore:

- Adding *neutrality*, we get *uniform* quota rules.
- Adding *unanimity* and *groundedness*, we get *nontrivial* rules.

F. Dietrich and C. List. Judgment Aggregation by Quota Rules: Majority Voting Generalized. *Journal of Theoretical Politics*, 19(4):391–424, 2007.

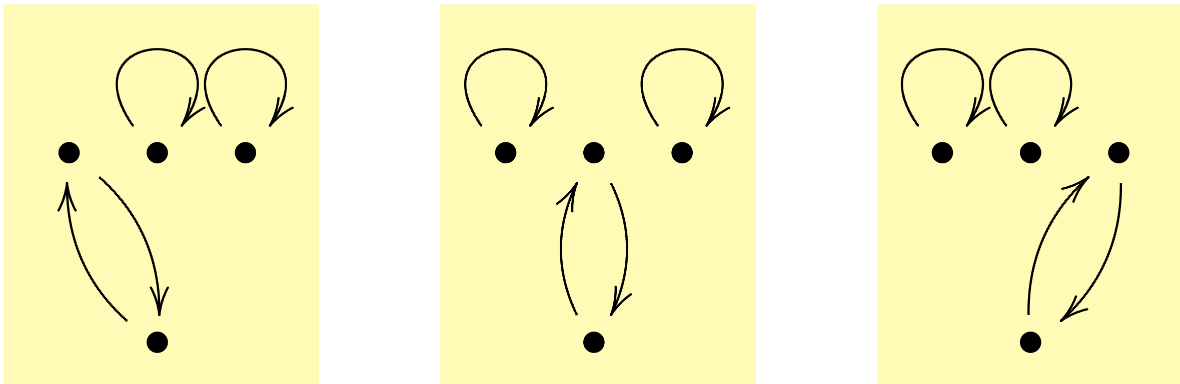
Collective Rationality

Aggregator F is *collectively rational* (CR) for graph property P if, whenever all individual graphs E_i satisfy P, so does the outcome $F(\boldsymbol{E})$.

Examples for *graph properties:* reflexivity, transitivity, seriality, ...

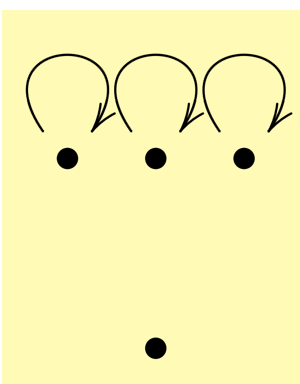
Example

Three agents each provide a graph on the same set of four vertices:



1 2 3

If we aggregate using the *majority rule*, we obtain this graph:



Observations:

- Majority rule not collectively rational for *seriality*.
- But *symmetry* is preserved.
- So is *reflexivity* (easy: individuals violate it).

Two Simple Possibility Results

The fact that the example worked for reflexivity is no coincidence:

Proposition 2 *Any unanimous aggregator is CR for reflexivity.*

Proof: If every individual graph includes edge (x, x), then unanimity ensures the same for the collective outcome graph. ✓

By a similar argument, we obtain:

Proposition 3 *Any grounded aggregator is CR for irreflexivity.*

Recall: Arrow's Theorem

This is how we had phrased Arrow's Theorem on Tuesday:

Theorem 4 (Arrow, 1951) *Any SWF for $\geqslant 3$ alternatives that satisfies the Pareto condition and IIA must be a dictatorship.*

This is the version for strict linear orders (Arrow's original formulation was for weak orders, which doesn't make much of a difference though).

I still owe you a proof.

K.J. Arrow. *Social Choice and Individual Values*. John Wiley and Sons, 2nd edition, 1963. First edition published in 1951.

Arrow's Theorem in Graph Aggregation

Our formulation in graph aggregation:

> *For $|V| \geqslant 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for irreflexivity, transitivity, and completeness.*

This implies the standard formulation, because:

- preferences (linear orders) = irreflexive, transitive, complete graphs
- nondictatorial = NR-nondictatorial for irreflexive graphs
- (weak) Pareto $\Rightarrow$ unanimous + grounded
- CR for irreflexivity is vacuous (implied by groundedness)

Main question for the next part:

▶ For what other classes of graphs does this go through?

Winning Coalitions

If an aggregator F is *independent*, then for every edge e there exists a set of *winning coalitions* $\mathcal{W}_e \subseteq 2^{\mathcal{N}}$ such that $e \in F(\boldsymbol{E}) \Leftrightarrow N_e^{\boldsymbol{E}} \in \mathcal{W}_e$.

Furthermore:

- If F is *unanimous*, then $\mathcal{N} \in \mathcal{W}_e$ for all edges e.
- If F is *grounded*, then $\emptyset \notin \mathcal{W}_e$ for all edges e.
- If F is *neutral*, then there is *one* $\mathcal{W}$ with $\mathcal{W} = \mathcal{W}_e$ for all edges e.

Proof Plan

Given: Arrovian aggregator F (*unanimous*, *grounded*, *independent*)
Want: Impossibility for *collective rationality* for graph property P

This will work if P is *contagious*, *implicative*, and *disjunctive* (TBD).

Lemma: CR for *contagious* $P \Rightarrow F$ is NR-*neutral*.

$\Rightarrow F$ characterised by some $\mathcal{W}$: $(x, y) \in F(\boldsymbol{E}) \Leftrightarrow N^{\boldsymbol{E}}_{(x,y)} \in \mathcal{W}$ $[x \neq y]$

Lemma: CR for *implicative* & *disjunctive* $P \Rightarrow \mathcal{W}$ is an *ultrafilter*, i.e.:

(i) $\emptyset \notin \mathcal{W}$ [this is immediate from groundedness]
(ii) $C_1, C_2 \in \mathcal{W}$ implies $C_1 \cap C_2 \in \mathcal{W}$ (closure under intersections)
(iii) C or $\mathcal{N} \setminus C$ is in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$ (maximality)

$\mathcal{N}$ is *finite* $\Rightarrow \mathcal{W}$ is *principal*: $\exists i^\star \in \mathcal{N}$ s.t. $\mathcal{W} = \{C \in 2^{\mathcal{N}} \mid i^\star \in C\}$
But this just means that $i^\star$ is a dictator: F is (NR-)*dictatorial*. ✓

Neutrality Lemma

Consider any Arrovian aggregator (unanimous, grounded, independent).

Call a property P *xy/zw-contagious* if there exist disjoint $S^+, S^- \subseteq V \times V$ s.t. every graph $E \in P$ satisfies $[\bigwedge S^+ \wedge \neg \bigvee S^-] \to [xEy \to zEw]$.

CR for xy/zw-contagious P implies: coalition $C \in \mathcal{W}_{(x,y)} \Rightarrow C \in \mathcal{W}_{(z,w)}$

Call P *contagious* if it satisfies (at least) one of the three conditions below:

(i) P is xy/yz-contagious for all $x, y, z \in V$.
(ii) P is xy/zx-contagious for all $x, y, z \in V$.
(iii) P is xy/xz-contagious and xy/zy-contagious for all $x, y, z \in V$.

Example: *Transitivity* ($[yEz] \to [xEy \to xEz]$ and $[zEx] \to [xEy \to zEy]$)

Contagiousness allows us to reach every NR edge from every other NR edge. Thus, *CR for contagious P* implies $\mathcal{W}_e = \mathcal{W}_{e'}$ for all NR edges e, e'.

So: *Collective rationality* for a *contagious* property implies NR-*neutrality*.

Ultrafilter Lemma

Let F be *unanimous*, *grounded*, *independent*, NR-*neutral*, and *CR* for P. So there exists a family of winning coalitions $\mathcal{W}$ s.t. $e \in F(\boldsymbol{E}) \Leftrightarrow N_e^{\boldsymbol{E}} \in \mathcal{W}$.

Show that $\mathcal{W}$ is an *ultrafilter* (under certain assumptions on P):

(i) $\emptyset \not\in \mathcal{W}$: immediate form groundedness

(ii) *Closure under intersections:* $C_1, C_2 \in \mathcal{W} \Rightarrow C_1 \cap C_2 \in \mathcal{W}$

Call P *implicative* if there exist disjoint sets $S^+, S^- \subseteq V \times V$ and distinct edges $e_1, e_2, e_3 \in V \times V \setminus (S^+ \cup S^-)$ s.t. all graphs $E \in P$ satisfy $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \wedge e_2 \rightarrow e_3]$.

Example: transitivity

CR for implicative $P \Rightarrow$ closure under intersections

Proof: Consider a profile where C_1 accept e_1, C_2 accept e_2, $C_1 \cap C_2$ accept e_3, everyone accepts S^+, and nobody accepts any edge in S^-. ✓

Ultrafilter Lemma (continued)

Still showing that $\mathcal{W}$ is an *ultrafilter* (for certain assumptions on P):

(iii) *Maximality:* C or $\mathcal{N} \setminus C$ in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$

Call P *disjunctive* if there exist disjoint sets $S^+, S^- \subseteq V \times V$ and distinct edges $e_1, e_2 \in V \times V \setminus (S^+ \cup S^-)$ s.t. all graphs $E \in P$ satisfy $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \vee e_2]$.

Example: completeness

CR for disjunctive $P \Rightarrow$ maximality

Proof: Consider a profile where C accept e_1, $\mathcal{N} \setminus C$ accept e_2, everyone accepts S^+, and nobody accepts any edge in S^-. ✓

End of Proof: Dictatorship

We have shown that our assumptions imply that F is characterised by a *single family* $\mathcal{W}$ of winning coalitions ($(x,y) \in F(\boldsymbol{E}) \Leftrightarrow N^{\boldsymbol{E}}_{(x,y)} \in \mathcal{W}$ for $x \neq y$) and that $\mathcal{W}$ must be an *ultrafilter:*

(i) $\emptyset \notin \mathcal{W}$

(ii) $C_1, C_2 \in \mathcal{W}$ implies $C_1 \cap C_2 \in \mathcal{W}$ (closure under intersections)

(iii) C or $\mathcal{N} \setminus C$ is in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$ (maximality)

Take the *intersection* of all winning coalitions (possible, as $\mathcal{N}$ is *finite*).
By (ii), this must be a winning coalition itself.
By (i), not empty. By (iii) cannot have two or more elements.
Thus, it must be a singleton $\{i^\star\}$, meaning that $i^\star$ is a dictator. ✓

General Impossibility Theorem

We have seen a proof for the following theorem:

Theorem 5 *For $|V| \geqslant 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for any graph property that is contagious, implicative, and disjunctive.*

Many combinations of graph properties have our meta-properties:

		C/I/D
Transitivity	$\forall xyz.(xEy \wedge yEz \rightarrow xEz)$	$++-$
Right Euclidean	$\forall xyz.(xEy \wedge xEz \rightarrow yEz)$	$++-$
Left Euclidean	$\forall xyz.(xEy \wedge zEy \rightarrow zEx)$	$++-$
Seriality	$\forall x.\exists y.xEy$	$--+$
Completeness	$\forall xy.[x \neq y \rightarrow (xEy \vee yEx)]$	$--+$
Connectedness	$\forall xyz.[xEy \wedge xEz \rightarrow (yEz \vee zEy)]$	$+++$
Negative Transitivity	$\forall xyz.[xEy \rightarrow (xEz \vee zEy)]$	$+-+$

Arrow's Theorem: use transitivity and completeness ✓

Collective Rationality and Modal Logic

Modal logic is a useful language for talking about graphs. This suggests trying to express CR requirements in modal logic. On the following slides, we will see some preliminary results in this directions:

- The modal logic perspective suggests a differentiation into three *levels of collective rationality*.
- For properties expressible as modal logic formulas satisfying certain *syntactic constraints*, we obtain simple *possibility results*.

I shall assume familiarity with basic modal logic.

Levels of Collective Rationality

Graphs $\langle V, E \rangle$ may be considered Kripke frames. The semantics of modal logic suggests three levels of collective rationality:

- F is *frame-CR* for a modal integrity constraint φ if $\langle V, E_i \rangle \models \varphi$ for all $i \in \mathcal{N}$ implies $\langle V, F(\boldsymbol{E}) \rangle \models \varphi$.
- F is *model-CR* for a modal IC φ if for all valuations $Val : \Phi \to 2^V$ $\langle \langle V, E_i \rangle, Val \rangle \models \varphi$ for all $i \in \mathcal{N}$ implies $\langle \langle V, F(\boldsymbol{E}) \rangle, Val \rangle \models \varphi$.
- F is *world-CR* for a modal IC φ if for all valuations $Val : \Phi \to 2^V$ and worlds $x \in V$ we have $\langle \langle V, E_i \rangle, Val \rangle, x \models \varphi$ for all $i \in \mathcal{N}$ implying $\langle \langle V, F(\boldsymbol{E}) \rangle, Val \rangle, x \models \varphi$.

Via modal correspondence theory, frame-CR corresponds to our original notion of collective rationality.

Connections

Proposition 6 *Let F be an aggregator and let φ a modal integrity constraint. Then the following implications hold:*

(i) *If F is world-CR for φ, then F is also model-CR for φ.*

(ii) *If F is model-CR for φ, then F is also frame-CR for φ.*

These implications are strict. Example:

> Suppose F returns the full graph if all individual graphs satisfy $\Diamond(p \vee \neg p)$, and the empty graph otherwise. Then F is model-CR but not world-CR for $\Diamond(p \vee \neg p)$: Take a profile of graphs with two worlds where $E_i = \{(x, y)\}$ for all $i \in \mathcal{N}$. The outcome returned by F is the empty graph, in violation of world-CR for $\Diamond(p \vee \neg p)$ at world x.

Remark: Impossibility results are most interesting for frame-CR. Possibility results are most interesting for world-CR.

Possibility Results

Let us call a *$\Box$-formula* any formula in NNF without any occurrences of $\Diamond$ (and define *$\Diamond$-formulas* accordingly).

Proposition 7 *If an aggregator F is such that for every profile $\boldsymbol{E}$ there exists an individual $i^\star \in \mathcal{N}$ such that $F(\boldsymbol{E}) \subseteq E_{i^\star}$, then F is world-CR for all $\Box$-formulas.*

Proposition 8 *If an aggregator F is such that for every profile $\boldsymbol{E}$ there exists an individual $i^\star \in \mathcal{N}$ such that $F(\boldsymbol{E}) \supseteq E_{i^\star}$, then F is world-CR for all $\Diamond$-formulas.*

Proposition 9 *If an aggregator F is such that for every profile $\boldsymbol{E}$ there exists an individual $i^\star \in \mathcal{N}$ such that $F(\boldsymbol{E}) = E_{i^\star}$, then F is world-CR for all modal integrity constraints.*

This last result is related to the fact that no representative-voter rule can ever cause a paradox (lecture on binary aggregation).

Summary

We have introduced *graph aggregation* as a generalisation of preference aggregation and then considered *collective rationality*.

Why is this interesting?

- Potential for *applications:* abstract argumentation, social networks
- Deep insights into the *structure of impossibilities:* direct link between CR requirements and neutrality/ultrafilter conditions

Topics covered:

- Axiomatic characterisation of quota rules
- Simple possibility results (e.g., unanimity lifting reflexivity)
- General impossibility theorem, ultrafilter proof technique
- The modal logic perspective