Voting Theory

Voting Theory AAAI-2010

Tutorial on Voting Theory

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Introduction

Voting theory (which is part of Social Choice Theory) is the study of methods for conducting an election:

- A group of voters each have preferences over a set of alternatives.

Each voter submits a ballot, based on which a voting procedure selects a (set of) winner(s) from amongst the alternatives.

This is not a trivial problem. Remember Florida 2000:

- 49%: Bush ≻ Gore ≻ Nader
- 20%: Gore ≻ Nader ≻ Bush
- 20%: Gore ≻ Bush ≻ Nader
- 11%: Nader ≻ Gore ≻ Bush

Voting Theory and AI

Voting theory has natural applications in AI:

- • Search Engines: to determine the most important sites based on links ("votes") + to aggregate the output of several search engines
- • Recommender Systems: to recommend a product to a user based on earlier ratings by other users
- • Multiagent Systems: to coordinate the actions of groups of autonomous software agents
- • AI Competitions: to determine who has developed the best trading agent / SAT solver / RoboCup team

But not all of the classical assumptions will fit these new applications. So AI needs to develop new models of voting and ask new questions.

Voting Theory and AI

Vice versa, techniques from AI, and computational techniques in general, are useful for advancing the state of the art in voting theory:

- • Algorithms and Complexity: to develop algorithms for (complex) voting procedures + to understand the hardness of "using" them
- • Knowledge Representation: to compactly represent the preferences of voters over large spaces of alternatives
- • Logic and Automated Reasoning: to formally model problems in voting theory + to automatically verify (or discover) theorems

Indeed, you will find many papers on voting at AI conferences (e.g., IJCAI, AAAI, AAMAS) and many AI researchers participate in events dedicated to voting and social choice (particularly COMSOC).
resolute candidates scoring vector voting correspondences simple majority, each voter initially votes for one (wins any pairwise contest).

Borda count voting procedures) beats B positional scoring rule beats B is the positional scoring rule with the scoring plurality rule (or plurality rule with run-off is the sum of all the points. Absolute majority. Tie-breaking is a separate issue.

Remarks: Each voter submits a ballot showing the name, and how to each voter submits a complete ranking of all alternatives. The alternative with the highest score (sum of points) wins. Remarks: The procedure defines what are •aggregate alternatives, e.g., the name of a single candidate is perceived to have little chance of winning. •Plurality with run-off: Gore wins (Nader eliminated in round 1) •Borda: Gore wins (Bush eliminated in round 1) •Gore is also the Condorcet winner (or most Condorcet winner) Alternative, a ranking of all alternatives, or something else.

We can formalise this idea introducing the Borda rule: 

\[ B = \left( s_1, s_2, \ldots, s_m \right) \]

Formally, \( B \) is given by a Borda rule

\[ B = \left( s_1, s_2, \ldots, s_m \right) \]

\[ s_i = \begin{cases} 0 & \text{if voter } i \text{ places } s \text{ at the } m \text{th position.} \\ i & \text{if voter } i \text{ places } s \text{ at the } i \text{th position.} \\ m & \text{if voter } i \text{ does not vote for } s. \end{cases} \]

The candidate with the highest Borda count wins.

Each voter submits a ranking of the candidates. Each voter submits a complete ranking of all alternatives. Each voter votes by submitting a ballot showing the name of a single candidate. The alternative with the highest score (sum of points) wins.

Plurality with run-off: Gore wins (Nader eliminated in round 1) Borda: Gore wins (Bush eliminated in round 1) Gore is also the Condorcet winner (or most Condorcet winner) Alternative, a ranking of all alternatives, or something else.

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The candidate with the highest Borda count wins.
Example

Positional Scoring Rules where Condorcet

\[
\begin{array}{c|ccc}
\text{ } & A & B & C \\
\hline
A & 0 & 1 & 1 \\
B & 0 & 0 & 2 \\
C & 0 & 1 & 0 \\
\end{array}
\]

Consider the following example:

\[A \succ B \succ C\]

\[C \succ A \succ B\]

\[B \succ C \succ A\]

There may be no Condorcet winner, witness the Condorcet paradox:

\[A \succ B \succ C\]

\[B \succ C \succ A\]

\[C \succ A \succ B\]

The Condorcet Principle
In approval voting, a ballot is a set of alternatives (the ones the voter "approves" of). The alternative with the most approvals wins.

Remarks:
- Approval voting has been used by several professional societies, such as the American Mathematical Society (AMS).
- Intuitively, less cause to vote for the most preferred candidate for strategic reasons when she has a slim chance of winning.
- A good compromise between plurality (too simple) and Borda (too complex) in terms of communication requirements.
- Only procedure we have seen where ballots cannot be modelled as linear orders over the set of alternatives.


We have seen a fair number of voting procedures:
- Ballots might be elements (plurality), rankings (e.g., Borda), or subsets (approval) of the set of alternatives.
- Types of procedures:
  - Positional scoring rules: Borda, plurality
  - Based on the majority graph: Copeland, voting trees
  - Based on the weighted majority graph: Kemeny
  - Staged procedures: plurality with run-off, STV

We have seen a few properties of voting procedures:
- Monotonicity, as violated by e.g. the no-show paradox
- Strategic issues, meaning people might not vote truthfully
- Condorcet principle: if an alternative wins all pairwise majority contests, then it should win the election

There are many more voting procedures, including:
- Antiplurality (or Veto) Rule, the Banks Rule, Black's Rule, Bucklin Voting, the Condorcet Rule, Coombs' Method, (General) Cumulative Voting, the Dodgson Rule, Even-and-Equal Cumulative Voting, Majority Judgment, Maximin Voting, Range Voting, the Slater Rule, and Tideman's Procedure.

Most textbooks on Social Choice Theory (some to be cited later) will introduce at least a small number of voting procedures. This first part of the tutorial owes much to the handbook chapter of Brams and Fishburn (2002), who discuss a good number of voting procedures in detail (with a certain emphasis on approval voting).

Nurmi (1987) devotes an entire book to the analysis of the properties of different voting procedures.


The axiomatic method:
- Most of the important classical results in voting theory are axiomatic. They formalise desirable properties as "axioms" and then establish:
  - Characterisation Theorems, showing that a particular (class of) procedure(s) is the only one satisfying a given set of axioms
  - Impossibility Theorems, showing that there exists no voting procedure satisfying a given set of axioms

We will see two examples each (and one other thing).
Anonymity and neutrality; A voting procedure satisfies the property of anonymity if
whenever some voter raises a (possibly tied) winner $x$ in her ballot, then $x$ will become the unique winner.

Formally:

For a full formalisation of this concept we would need to be able to speak about a voting procedure wrt. different electorates $N$ and any permutation $\pi$ of $N$.

Whenever

there exists a

anonymity

contradiction.

Every scoring vector $s$ defines a PSR: give $s$ points to alternative $i$ at the $i$th position; the winners are the $i$ alternatives with the most points.

A voting rule is continuous if, whenever we split the electorate into two groups and some alternative would win in both, then only the number of votes matters. By PR, now only one alternative wins. Then, by PR, now only one alternative wins.

A voting procedure satisfies positive responsiveness if
whenever someone ranks alternatives with the most points.


Reinforcement (a.k.a. Consistency)

Neutrality

If, and only if, whenever we split the electorate into two groups and some alternative would win in both, then only the number of votes matters.

May's Theorem


A voting procedure is

only number of votes matters.

Proof Sketch

Maybe the best known result of this kind is Young's characterisation of continuous characterisation theorem. It states that there exist two cases:

- There exist two voters switch ballots, then the winners don't change.
- For a full formalisation of this concept we would need to be able to speak about a voting procedure wrt. different electorates $N$ and any permutation $\pi$ of $N$.

Whenever

there exist

anonymity
Under the assumption of a non-empty set of alternatives, an important property of a social welfare function is independence of irrelevant alternatives (IIA). This means that if the relative ranking of two alternatives remains the same, then the ranking of the remaining alternatives should not be affected. Formally, if \( x \succ y \) for all profiles \( b \in B \), then \( F(\{x, y\}) = F(\{y, x\}) \) for all profiles \( b \in B \).

Theorem 3 (Arrow, 1951) Theorem 4 (Young, 1975)

A voting procedure is nondictatorial if and only if it is IIA. This variant of IIA (for voting rules) is due to Taylor (2005). This is widely regarded as the most debatable of the three axioms featuring in the impossibility theorems (and difficult).
The median voter rule is strategy-proof (Arrow fails). If there exist no profile single-peaked alternatives that is Condorcet winner.

The main steps are: observe that, by definition, there can be only one dictator can dictate their relative position wrt. any y. As x is between y and her top alternative wrt. x, by one, there must be a x is the median, for more than half of the voters

**Black's Theorem (Black's Theorem, 1948)**

If an odd number of voters submit their favourite, so they prefer x to, say, y and their favourite, so they prefer x to, say, z. As x is between y and z, x must be elected by the voter corresponding to the median wrt. y and her top alternative wrt. x. As x is between y and z, x must be elected by the voter corresponding to the median wrt. y and her top alternative wrt. x. Hence, Black's Median Voter Theorem: A Simple Proof Theorem 5 (Black's Theorem, 1948)

**Gibbard-Satterthwaite Theorem (Gibbard-Satterthwaite, 1973)**

Any strategy-proof election procedure (Gibbard-Satterthwaite fails).

**Consequences**

• If the number of voters is odd and there are at least three alternatives, then the median voter rule is strategy-proof.

• The Gibbard-Satterthwaite Theorem is a singleton for any profile of ballots.

• The median voter rule (= electing the Condorcet winner) is strategy-proof.

**Remarks:**

• The same definition can be applied to profiles of ballots.

• Note that we have made an implicit linear order may come up as a preference or ballot.

• Any weakly Pareto efficient election procedure is strategy-proof. How can we circumvent these impossibilities?

• For simplicity, assume the number of voters is n, . . . , b, n, . . . , b.
During the remainder of the tutorial, we will see some examples of how logic has been used to formalise specific computational systems and to facilitate formal or even computational problem solving. Typically, these methods can be applied to problems in voting and social choice.

The kind of research can be broadly classified along two dimensions —

1. Logical and Automated Reasoning
   - What logic fits best?
   - Which automated reasoning methods are useful?
   - How can we apply this methodology also to social choice mechanisms?

2. Computational Social Choice
   -Which automated reasoning methods are useful?
   - How can we apply this methodology also to social choice mechanisms?
   - What logic fits best?

Examples

Logic has long been used to formalise specific computational systems and to facilitate formal or even computational problem solving. Typically, these methods can be applied to problems in voting and social choice.

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   - What logic fits best?

Summary: Major Theorems

Two classics to look out for:

- Arrow's impossibility of a Pareto efficient, non manipulative, non-dictatorial voting system.
- Gibbard-Satterthwaite theorem: any majority graph can occur.


58 paper submissions and 80 participants (14 countries).

Other classics to look out for:

- Black's single-peakedness solves most problems.
- Arrow and Sen: triple-wise value restrictions and scoring single-peakedness.
- Sen: triple-wise value restrictions and scoring single-peakedness.
- Groves: strategy-proofness for quasi-linear preferences.
- Duggan-Schwartz: strategy-proofness leads to dictatorships.
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- Groves: strategy-proofness leads to dictatorships.
The Borda rule is also easy to manipulate. Use a hypothesis regarding new impossibility theorems. Recall Borda: submit a ranking (super-polynomial many choices!) and for the complexity verify Borda rule is for computing the winners of complex voting procedures.

Recall STV: eliminate plurality losers until an alternative gets a plurality of the votes.

If the theorem in the HOL proof assistant Isabelle is correct, we have NP-membership is clear: checking whether a given ballot makes the alternative to be made winner by means of STV is in the complexity class NP.

Otherwise not.

Theorem in the HOL proof assistant Isabelle (2010):

NP-complete.

Manipulation of STV is Theorem 6 (Bartholdi and Orlin, 1991): the manipulation of STV is NP-complete. Next: The seminal paper by Bartholdi, Tovey and Trick (1989) starts by showing that manipulation is in fact easy.

For a range of commonly used rules we are able to prove NP-hardness.

We then present a result from a follow-up paper by Bartholdi and Orlin (1991):

Bartholdi and Orlin (1991): the manipulation of STV is NP-complete.


decision problem: Manipulating an Election.

We first present a couple of these easiness results, namely for each voter submits a ballot showing the name of one of the alternatives. The alternative receiving the most votes wins.


decision problem: Manipulating the Plurality Rule

We need to show NP-hardness and NP-membership.

Each voter submits a ballot showing the name of one of the alternatives, in order of her preference.

Then inductively proceed as follows: Check if any of the remaining alternatives can be put next into the preference ordering without preventing the alternative to be made winner by means of STV. If yes, do so. If no, terminate and say that manipulation is impossible.

Question: Is there a ballot for the final voter such that manipulability may be considered less of a worry for procedure with a particular number of alternatives?

If there exists an Arrovian aggregator for $n$ alternatives, then there exists one for $m$ alternatives, for all $m > n$. This is shown in the seminal paper by Bartholdi, Tovey and Trick (1989).

If $m > n$, then there exists one for $n$ alternatives.

If no, terminate and say that manipulation is impossible.

Let $x$ be a ballot of $n$ voters.

For all alternatives, in order of her preference.

After convincing ourselves that this algorithm is indeed correct, we are able to prove NP-hardness.

Discussion:

NP-hardness for a range of commonly used rules.

The alternative to be made winner by means of STV if and only if it has $50\%$ of the votes.

Computing the winners for a range of voting procedures can be done in polynomial time.

Then inductively proceed as follows: Check if any of the remaining alternatives can be put next into the preference ordering without preventing the alternative to be made winner by means of STV. If yes, do so. Otherwise not.

Discussion:

NP-hardness for a range of voting procedures can be done in polynomial time.

Then inductively proceed as follows: Check if any of the remaining alternatives can be put next into the preference ordering without preventing the alternative to be made winner by means of STV. If yes, do so. Otherwise not.


decision problem: Manipulating the Borda Rule

A public choice of any of the alternatives can be done in polynomial time.

Computing the winners for a range of voting procedures can be done in polynomial time.

Then inductively proceed as follows: Check if any of the remaining alternatives can be put next into the preference ordering without preventing the alternative to be made winner by means of STV. If yes, do so. Otherwise not.


decision problem: Manipulating the Plurality Rule

NP-hardness for a range of voting procedures can be done in polynomial time.
The text seems to be discussing voting theory and combinatorial votes. It mentions the Borda rule, paradox of multiple elections, and other voting rules and procedures. The text also talks about the succinctly encoded ballots received and how complex it is to reason about this information. It mentions the winner of an election and the Hamming distance to any of the ballots. There are examples and solutions provided, as well as discussion on further approaches and more on complexity of voting.
We have seen a small selection of samples of COMSOC research:

- Logic and automated reasoning for verification of results in SCT (also interesting: formalisation, discovery)
- Complexity theory to distinguish possibility from feasibility (for manipulation, winner determination, and more)
- KR for modelling social choice in combinatorial domains

There is a growing COMSOC research community out there, investigating these issues and much more:

- other questions in voting and preference aggregation
- fair division, stable matchings, judgment aggregation, ...

Chevaleyre et al. (2007) classify contributions in COMSOC wrt. the computational method used and the social choice problem addressed.

Faliszewski and Procaccia (2010) review work on the complexity of manipulation (the archetypical COMSOC problem).

Chevaleyre et al. (2008) give an introduction to social choice in combinatorial domains.


Nice topic, particularly for AI people. Still lots to do.

A website where you can find out more about Computational Social Choice:

http://www.illc.uva.nl/COMSOC/

These slides will remain available on the tutorial website, and more extensive materials can be found on the website of my Amsterdam course on Computational Social Choice:

– http://www.illc.uva.nl/~ulle/teaching/comsoc/