# Computational Social Choice: Spring 2007

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# Plan for Today

This lecture will be an introduction to voting theory. Voting is the most obvious mechanism by which to come to a collective decision, so it is a central topic in social choice theory. Topics today:

- many *voting procedures:* e.g. plurality rule, Borda count, approval voting, single transferable vote, ...
- several (desirable) *properties* of voting procedures: e.g. anonymity, neutrality, monotonicity, strategy-proofness, ...
- some voting *paradoxes*, highlighting that there seems to be no perfect voting procedure

Most of the material on these slides is taken from a review article by Brams and Fishburn (2002).

S.J. Brams and P.C. Fishburn. *Voting Procedures.* In K.J. Arrow *et al.* (eds.), *Handbook of Social Choice and Welfare*, Elsevier, 2002.

# Voting Rules

- We'll discuss *voting rules* for selecting a *single winner* from a finite set of *candidates*. (The number of candidates is $m$.)
- A *voter* votes by submitting a *ballot*. This could be the name of a single candidate, a complete ranking of all the candidates, or something else.
- A voting rule has to specify what makes a *valid ballot*, and how the preferences expressed via the ballots are to be *aggregated* to produce the election winner.
- All of the voting rules to be discussed allow for the possibility that two or more candidates come out on top (although this is unlikely for large numbers of voters). A complete system would also have to specify how to deal with such ties, but here we are going to ignore the issue of *tie-breaking*.

# Plurality Rule

Under the *plurality rule* (a.k.a. *simple majority*), each voter submits a ballot showing the name of one of the candidates standing. The candidate receiving the most votes wins.

This is the most widely used voting rule in practice.

Problems with the plurality rule include:

- Dispersion of votes across ideologically similar candidates ($\leadsto$ extremist candidates, negative campaigning).
- Encourages voters not to vote for their true favourite, if that candidate is perceived to have little chance of winning.

## Plurality with Run-Off

In the *plurality rule with run-off*, first each voter votes for one candidate. The winner is elected in a second round by using the plurality rule with the two top candidates from the first round.

Used to elect the president in France (and heavily criticised after Le Pen came in second in the first round in 2002).

# The No-Show Paradox

Under plurality with run-off, it may be better to abstain than to vote for your favourite candidate! Example:

$$\begin{array}{ll} \text{25 voters:} & A \succ B \succ C \\ \text{46 voters:} & C \succ A \succ B \\ \text{24 voters:} & B \succ C \succ A \end{array}$$

Given these voter preferences, $B$ gets eliminated in the first round, and $C$ beats $A$ 70:25 in the run-off.

Now suppose two voters from the first group abstain:

$$\begin{array}{ll} \text{23 voters:} & A \succ B \succ C \\ \text{46 voters:} & C \succ A \succ B \\ \text{24 voters:} & B \succ C \succ A \end{array}$$

$A$ gets eliminated, and $B$ beats $C$ 47:46 in the run-off.

# Monotonicity

We would like a voting rule to satisfy *monotonicity:* if a particular candidate wins and a voter raises that candidate in their ballot (whatever that means exactly for different sorts of ballots), then that candidate should still win.

The *winner-turns-loser paradox* shows that plurality with run-off does *not* satisfy monotonicity:

$$\begin{array}{ll} \text{27 voters:} & A \succ B \succ C \\ \text{42 voters:} & C \succ A \succ B \\ \text{24 voters:} & B \succ C \succ A \end{array}$$

$B$ gets eliminated in the first round and $C$ beats $A$ 66:27 in the run-off. But if 4 of the voters from the first group *raise $C$ to the top* (*i.e.* join the second group), then $B$ will win (it's the same example as on the previous slide).

# Anonymity and Neutrality

On the positive side, both variants of the plurality rule satisfy two important properties:

- *Anonymity:* A voting rule is anonymous if it treats all *voters* the same: if two voters switch ballots the election outcome does not change.
- *Neutrality:* A voting rule is neutral if it treats all *candidates* the same: if the election winner switches names with some other candidate, then that other candidate will win.

Indeed, (almost) all of the voting rules we'll discuss satisfy these properties (we'll see one exception where neutrality is violated).

Often the *tie-breaking* rule can be a source of violating either anonymity (e.g. if one voter has the power to break ties) or neutrality (e.g. if the incumbent wins in case of a tie).

# Borda Rule

Under the voting rule proposed by Jean-Charles de Borda, each voter submits a complete ranking of all the $m$ candidates.

For each voter that places a candidate first, that candidate receives $m-1$ points, for each voter that place her 2nd she receives $m-2$ points, and so forth. The *Borda count* is the sum of all the points.

The candidate with the highest Borda count wins.

This takes care of some of the problems we have identified for plurality voting. For instance, the Borda rule satisfies monotonicity.

A disadvantage (of any system requiring voters to submit full rankings) are the high *elicitation* and *communication* costs.

J.-C. de Borda. *Mémoire sur les élections au scrutin.* Histoire de l'Académie Royale des Sciences, Paris, 1781.

## Pareto Principle

A voting rule satisfies the *Pareto principle* if, whenever candidate $A$ is preferred over candidate $B$ by all voters (and strictly preferred by at least one), then $B$ cannot win the election.

Clearly, both the plurality rule and the Borda rule satisfy the Pareto principle.

# Positional Scoring Rules

We can generalise the idea underlying the Borda count as follows:

Let $m$ be the number of candidates. A *positional scoring rule* is given by a *scoring vector* $s = \langle s_1, \ldots, s_m \rangle$ with $s_1 \geq s_2 \geq \cdots \geq s_m$.

Each voter submits a ranking of all candidates. Each candidate receives $s_i$ points for every voter putting her at the $i$th position. The candidate with the highest score (sum of points) wins.

- The *Borda rule* is is the positional scoring rule with the scoring vector $\langle m-1, m-2, \ldots, 0 \rangle$.
- The *plurality rule* is the positional scoring rule with the scoring vector $\langle 1, 0, \ldots, 0 \rangle$.

# Condorcet Principle

Recall the Condorcet Paradox (first lecture):

$$\begin{array}{ll} \text{Voter 1:} & A \succ B \succ C \\ \text{Voter 2:} & B \succ C \succ A \\ \text{Voter 3:} & C \succ A \succ B \end{array}$$

A majority prefers $A$ over $B$ and a majority also prefers $B$ over $C$, but then again a majority prefers $C$ over $A$. Hence, no single candidate would beat any other candidate in pairwise comparisons.

In cases where the *is* such a candidate beating everyone else in a pairwise majority contest, we call her the *Condorcet winner*.

(Assuming that voter preferences are linear and the number of voters is odd, a Condorcet winner, if any, must be unique.)

A voting rule is said to satisfy the *Condorcet principle* if it elects the Condorcet winner whenever there is one.

# Positional Soring violates Condorcet

Consider the following example:

$$\begin{array}{ll} \text{3 voters:} & A \succ B \succ C \\ \text{2 voters:} & B \succ C \succ A \\ \text{1 voter:} & B \succ A \succ C \\ \text{1 voter:} & C \succ A \succ B \end{array}$$

*$A$* is the *Condorcet winner*; she beats both $B$ and $C$ 4:3. But any *positional scoring rule* assigning strictly more points to a candidate placed 2nd than to a candidate placed 3rd ($s_2 > s_3$) makes *$B$* win:

$$\begin{array}{ll} A: & 3 \cdot s_1 + 2 \cdot s_2 + 2 \cdot s_3 \\ B: & 3 \cdot s_1 + 3 \cdot s_2 + 1 \cdot s_3 \\ C: & 1 \cdot s_1 + 2 \cdot s_2 + 4 \cdot s_3 \end{array}$$

This shows that *no positional scoring rule* (with a strictly descending scoring vector) will satisfy the *Condorcet principle*.

# Copeland Rule

Some voting rules have been designed specifically to meet the Condorcet principle.

The *Copeland rule* elects a candidate that maximises the difference between won and lost pairwise majority contests.

The Copeland rule satisfies the Condorcet principle (as defined on these slides —if Condorcet winners may win *or draw* on majority contests, then there are counterexamples).

## Dodgson Rule

Charles L. Dodgson (a.k.a. Lewis Carroll) proposed a voting method that selects the candidate minimising the number of "switches" in the voters' linear preference orderings required to make that candidate a Condorcet winner.

Clearly, this metric is 0 if the candidate in question already is a Condorcet winner, so the *Dodgson rule* certainly satisfies the Condorcet principle.

C.L. Dodgson. *A Method of Taking Votes on more than two Issues.* Clarendon Press, Oxford, 1876.

# Approval Voting

In *approval voting*, a ballot may consist of any subset of the set of candidates. These are the candidates the voter approves of. The candidate receiving the most approvals wins.

Approval voting has been used by several professional societies, such as the American Mathematical Society (AMS).

Intuitive advantages of approval voting include:

- No need not to vote for a preferred candidate for strategic reasons, when that candidate has a slim chance to win (this is in fact *not true*, but at least the examples for successful manipulation are less obvious than for plurality voting).
- Seems like a good compromise between plurality (too simple) and Borda (too complex).

# Single Transferable Vote (STV)

Also known as the *Hare system*. To select a single winner, it works as follows (voters submit ranked preferences for all candidates):

- If one of the candidates is the 1st choice for over 50% of the voters (*quota*), she wins.
- Otherwise, the candidate who is ranked 1st by the fewest voters gets *eliminated* from the race.
- Votes for eliminated candidates get *transferred:* delete removed candidates from ballots and "shift" rankings (e.g. if your 1st choice got eliminated, then your 2nd choice becomes 1st).

In practice, voters need not be required to rank all candidates (non-ranked candidates are assumed to be ranked lowest).

STV (suitably generalised) is often used to elect committees.

STV is used in several countries (e.g. Australia, New Zealand, ...).

## Example

Elect one winner amongst four candidates, using STV (100 voters):

39 voters: $A \succ B \succ C \succ D$

20 voters: $B \succ A \succ C \succ D$

20 voters: $B \succ C \succ A \succ D$

11 voters: $C \succ B \succ A \succ D$

10 voters: $D \succ A \succ B \succ C$

(Answer: $B$ wins)

Note that for 3 candidates, STV reduces to plurality voting with run-off, so it suffers from the same paradoxes.

## Manipulation: Plurality Rule

Suppose the *plurality rule* (as in most real-world situations) is used to decide the outcome of an election.

Assume the preferences of the people in, say, Florida are as follows:

$$\begin{array}{ll} 49\%: & \text{Bush} \succ \text{Gore} \succ \text{Nader} \\ 20\%: & \text{Gore} \succ \text{Nader} \succ \text{Bush} \\ 20\%: & \text{Gore} \succ \text{Bush} \succ \text{Nader} \\ 11\%: & \text{Nader} \succ \text{Gore} \succ \text{Bush} \end{array}$$

So even if nobody is cheating, Bush will win in a plurality contest.

It would have been in the interest of the Nader supporters to *manipulate*, *i.e.* to misrepresent their preferences.

# The Gibbard-Satterthwaite Theorem

The Gibbard-Satterthwaite Theorem is widely regarded as *the* central result in voting theory. Broadly, it states that there can be no "reasonable" voting rule that would not be manipulable.

Our formal statement of the theorem follows Barberà (1983). We won't prove it here. A proof that is similar to the one we have discussed for Arrow's Theorem is given by Benoît (2000).

A. Gibbard. Manipulation of Voting Schemes: A General Result. *Econometrica*, 41(4):587–601, 1973.

M.A. Satterthwaite. Strategy-proofness and Arrow's Conditions. *Journal of Economic Theory*, 10:187–217, 1975.

S. Barberà. Strategy-proofness and Pivotal Voters: A Direct Proof of the Gibbard-Satterthwaite Theorem. *Intl. Economic Review*, 24(2):413–417, 1983.

J.-P. Benoît. The Gibbard-Satterthwaite Theorem: A Simple Proof. *Economic Letters*, 69:319–322, 2000.

# Setting and Notation

- Finite set $A$ of *candidates* (alternatives); finite set $I = \{1..n\}$ of *voters* (individuals).
- A *preference ordering* is a strict linear order on $A$. The set of all such orderings is denoted $\mathcal{P}$. Each voter $i$ has an *individual* preference ordering $P_i$. A *preference profile* $\langle P_1, \ldots, P_n \rangle \in \mathcal{P}^n$ consists of a preference ordering for each voter.
- The *top candidate* $top(P)$ of a preference ordering $P$ is defined as the unique $x \in A$ such that $xPy$ for all $y \in A \setminus \{x\}$.
- We write $(\boldsymbol{P}_{-i}, P')$ for the preference profile we obtain when we *replace* $P_i$ by $P'$ in the preference profile $\boldsymbol{P}$.
- A *voting rule* is a function $f : \mathcal{P}^n \rightarrow A$ mapping preference profiles to winning candidates (so the $P_i$ are used as ballots).

# Statement of the Theorem

A voting rule $f$ is *dictatorial* if the winner is always the top candidate of a particular voter (the dictator):

$$(\exists i \in I)(\forall \boldsymbol{P} \in \mathcal{P}^n)[f(\boldsymbol{P}) = top(P_i)]$$

A voting rule $f$ is *manipulable* if it may give a voter an incentive to misrepresent their preferences:

$$(\exists \boldsymbol{P} \in \mathcal{P}^n)(\exists P' \in \mathcal{P})(\exists i \in I)[f(\boldsymbol{P}_{-i}, P')\, P_i\, f(\boldsymbol{P})]$$

A voting rule that is not manipulable is also called *strategy-proof*.

**Theorem 1 (Gibbard-Satterthwaite)** *If $|A| > 2$, then every voting rule must be either dictatorial or manipulable.*

## Control: Borda Rule

The technical term "manipulation" refers to voters misrepresenting their preferences, but there are also other forms of manipulation ...

Suppose we are using the *Borda rule* to elect one winner from amongst 4 candidates, and there are 13 voters:

$$
\begin{array}{ll}
\text{4 voters:} & A \succ X \succ B \succ C \\
\text{3 voters:} & C \succ A \succ X \succ B \\
\text{6 voters:} & B \succ C \succ A \succ X
\end{array}
$$

We get the following Borda scores: $A$ (24), $B$ (22), $C$ (21), $X$ (11).

We may suspect the $A$-supporters of having nominated $X$ in order to *control* the election. For, without $X$, we get these Borda scores: $A$ (11), $B$ (16), $C$ (12).

This example also shows that the Borda rule is *not independent of irrelevant alternatives*.

# Agenda Manipulation: Voting Trees

The term *control* is used for any kind of "manipulation" that involves changing the structure of an election (voting rule, set of candidates, ...). This is typically something that the *election chair* may do (but not only; see nomination example on previous slide).

Consider the following example (Condorcet triple):

$$\begin{array}{ll} \text{Voter 1:} & A \succ B \succ C \\ \text{Voter 2:} & B \succ C \succ A \\ \text{Voter 3:} & C \succ A \succ B \end{array}$$

Suppose the voting rule is given by a *binary tree*, with the candidates labelling the leaves, and a candidate progressing to a parent node if beats its sibling in a *majority contest*.

Then the election chair can influence the election outcome by changing the *agenda* (here, the exact binary tree to be used) ...

# Agenda Manipulation: Voting Trees (cont.)

Here are again the voter preferences from the previous slide:

$$\begin{array}{ll} \text{Voter 1:} & A \succ B \succ C \\ \text{Voter 2:} & B \succ C \succ A \\ \text{Voter 3:} & C \succ A \succ B \end{array}$$

So in a pairwise majority contest, $A$ will beat $B$; $B$ will beat $C$; and $C$ will beat $A$. Here are two possible voting trees:

```
(1)                 (2)      o
                            / \
          o                /   \
         / \
        o   C             o     o
       / \               / \   / \
      A   B             A   B B   C
```

If `(1)` is used then $C$ will win; if `(2)` is used then $A$ will win. That is, these voting rules violate *neutrality*.

## Classification of Voting Procedures

Brams and Fishburn (2002) list many more voting procedures.

The structure of their paper implicitly suggests a (rough) classification of voting rules:

- Non-ranked: plurality rule, approval voting
- Non-ranked multi-stage: plurality with run-off, voting trees
- Condorcet procedures: Copeland, Dodgson, (many more)
- Positional scoring rules: Borda count

## Summary

This has been an introduction to voting theory. The main aim has been to show that there are many alternative systems, all with their own flaws and advantages.

- Voting procedures: plurality (with run-off), positional scoring rules, Condorcet procedures, approval, STV, voting trees, ...
- Properties discussed: anonymity, neutrality, monotonicity, Condorcet principle, strategy-proofness, ...
- Cheating can take many forms: manipulation, bribery, control
- Important technical result: Gibbard-Satterthwaite Theorem

Most of the material on these slides comes from (and much more can be found in) the review article by Brams and Fishburn (2002).

S.J. Brams and P.C. Fishburn. *Voting Procedures.* In K.J. Arrow *et al.* (eds.), *Handbook of Social Choice and Welfare*, Elsevier, 2002.

# What next?

This lecture has concentrated on classical topics in voting theory. Next week we are going to discuss *computational issues* in voting.

Two questions that suggest why this is of interest:

- What if we have found a voting rule with many wonderful theoretical properties, but actually *computing the winner* using that rule is a *computationally intractable* problem?
- What if *manipulation* is possible (by the Gibbard-Satterthwaite Theorem), but turns out to be *computationally intractable*, so no voter would ever be able to exploit this weakness?

The next class will include a very brief refresher of computational complexity theory (NP-completeness) ...