Introduction to Tournaments

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Notations

- $X$ is a finite set of alternatives.
- $T$ is a relation on $X$, i.e., $T \subset X^2$.
- notation: $(x, y) \in T \iff x Ty \iff x \rightarrow y \iff x \text{ "beats" } y$
- $\mathcal{T}(X)$ is the set of tournaments on $X$
- $T^+(x) = \{ y \in X \mid x Ty \}$: successors of $x$.
- $T^-(x) = \{ y \in X \mid y Tx \}$: predecessors of $x$.
- $s(x) = \#T^+(x)$ is the Copeland score of $x$.

- Voting
  
  Input: Preference of agents over a set of candidates or outcomes
  
  Output: one candidate or outcome (or a set)

- Tournament
  
  Input: Binary relation between outcomes or candidates
  
  Output: One candidate or outcome (or a set)

  When no ties are allowed between any two alternatives.
  Either $x$ beats $y$ or $y$ beats $x$.
  which are the best outcomes?

Definition (Tournament)

The relation $T$ is a tournament iff

- $\forall x \in X (x, x) \notin T$
- $\forall (x, y) \in X^2 x \neq y \Rightarrow [(x, y) \in T \lor (y, x) \in T]$
- $\forall (x, y) \in X^2 (x, y) \in T \Rightarrow (y, x) \notin T$.

A tournament is a complete and asymmetric binary relation.

Majority voting and tournament:

- $I$ finite set of individuals. The preference of an individual $i$ is represented by a complete order $P_i$ defined on $X$.
- The outcome of majority voting is the binary relation $M(P)$ on $X$ such that $\forall (x, y) \in X, x M(P) y \iff \# \{ i \in I \mid x P_i y \} > \# \{ i \in I \mid y P_i x \}$

If initial preferences are strict and number of individual is odd, $M(P)$ is a tournament.
Example (cyclone of order $n$)

$\mathbb{Z}_n$ set of integers modulo $n$.

$xC_n y \iff y - x \in \{1, \ldots, \frac{n-1}{2}\}$

$T^+(1) = \{2, 3, 4\}$

$T^-(1) = \{5, 6, 7\}$

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Definition (isomorphism)

Let $X$ and $Y$ be two sets, $T \in \mathcal{T}(X)$, $U \in \mathcal{T}(Y)$ two tournaments on $X$ and $Y$.

A mapping $\phi : X \to Y$ is a tournament isomorphism iff

- $\phi$ is a bijection
- $\forall (x, y) \in X^2, x T x' \iff \phi(x) U \phi(x')$

On a set $X$ of cardinal $n$, there are $2^{2^{n(n-1)/2}}$ tournaments, but many of them are isomorphic.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$2^{n(n-1)/2}$</th>
<th>number of non-isomorphic tournaments</th>
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<tbody>
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<td>35,184,372,088,832</td>
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Outline

1. Introduction: Reasoning about pairwise competition
2. Desirable properties of solution concepts
3. Solution based on scoring and Ranking
4. Solutions based on Covering
5. Solution based on Game Theory
6. Contestation Process
7. Knockout tournaments
8. Notes on the size of the choice set

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Condorcet principle

Definition (Condorcet winners)

Let $T \in \mathcal{T}(X)$. The set of Condorcet winners of $T$ is

$\text{Condorcet}(T) = \{x \in X \mid \forall y \in X, y \neq x \Rightarrow x T y\}$

Property

Either $\text{Condorcet}(T) = \emptyset$ or $\text{Condorcet}(T)$ is a singleton.
A tournament solution \( \mathcal{T} \) associates to any tournament \( \mathcal{T}(X) \) a subset \( \mathcal{T}(T) \subset X \) and satisfies

- \( \forall T \in \mathcal{T}(X), \mathcal{T}(T) \neq \emptyset \)
- For any tournament isomorphism \( \phi, \phi \circ \mathcal{T} = \mathcal{T} \circ \phi \) (anonymity)
- \( \forall T \in \mathcal{T}(X), \text{Condorcet}(T) \neq \emptyset \Rightarrow \mathcal{T}(T) = \text{Condorcet}(T) \)

For \( \mathcal{T}, \mathcal{T}_1, \mathcal{T}_2 \) tournament solutions,

- \( \mathcal{T}_1 \circ \mathcal{T}_2(T) = \mathcal{T}_1(T/\mathcal{T}_2(T)) = \mathcal{T}_1(\mathcal{T}_2(T)) \)
- \( \mathcal{T} = \mathcal{T}, \mathcal{T}^{k+1} = \mathcal{T}_0 \mathcal{T}^k, \mathcal{T}^\infty = \lim_{k \to \infty} \mathcal{T}^k \)

solutions may be finer/more selective:

- \( \mathcal{T}_1 \subset \mathcal{T}_2 \Leftrightarrow \forall T \in \mathcal{T}(X), \mathcal{T}_1(T) \subset \mathcal{T}_2(T) \)
- \( \mathcal{T}_1 \cap \mathcal{T}_2 \Leftrightarrow \forall T \in \mathcal{T}, \mathcal{T}_1(T) \cap \mathcal{T}_2(T) \neq \emptyset \)

Properties of Solutions

- Regular
- Monotonous
- Independent of the losers
- Strong Superset Property
- Idempotent
- Aizerman property
- Composition-consistent and weak composition-consistent

A first solution: the Top Cycle (TC)

Definition (Top Cycle)

The top cycle of \( T \in \mathcal{T}(X) \) is the set \( \text{TC}(T) \) defined as

\[
\text{TC}(T) = \left\{ x \in X \mid \forall y \in X, \exists k > 0 \text{ s.t. } \exists (z_1, \ldots, z_k) \in X^k, z_1 = x, z_k = y, 1 \leq i < j \leq k \Rightarrow z_i T z_j \right\}
\]

The top cycle contains outcomes that beat directly or indirectly every other outcomes.

Regular tournament

A tournament is regular iff all the points have the same Copeland score.

Monotonous

A solution \( \mathcal{T} \) is monotonous iff \( \forall T \in \mathcal{T}(X), \forall x \in \mathcal{T}(T), \forall T' \in \mathcal{T}(X) \) such that

\[
\begin{align*}
T'/X \setminus \{x\} &= T/X \setminus \{x\} \\
\forall y \in X, xTY &\Rightarrow xT'y
\end{align*}
\]

one has \( x \in \mathcal{T}(T') \).

“Whenever a winner is reinforced, it does not become a loser.”
Definition (Independence of the losers)
A solution $\mathcal{S}$ is independent of the losers iff $\forall T \in \mathcal{T}(X), \forall T' \in \mathcal{T}(X)$ such that $\forall x \in \mathcal{S}(T), \forall y \in X, xTy \iff xT'y$ one has $\mathcal{S}(T) = \mathcal{S}(T')$.

"the only important relations are \{ winners to winners \}
"What happens between losers do not matter."

Definition (Strong Superset Property (SSP))
A solution $\mathcal{S}$ satisfies the Strong Superset Property (SSP) iff $\forall T \in \mathcal{T}(X), \forall Y | \mathcal{S}(T) \subset Y \subset X$ one has $\mathcal{S}(T) = \mathcal{S}(T/Y)$

"We can delete some or all losers, and the set of winners does not change"

Definition (Idempotent)
A solution $\mathcal{S}$ is idempotent iff $\mathcal{S} \circ \mathcal{S} = \mathcal{S}$.

Solution Concepts
- Copeland solution (C)
- the Long Path (LP)
- Markov solution (MA)
- Slater solution (SL)
- Uncovered set (UC)
- Iterations of the Uncovered set ($UC^\infty$)
- Dutta’s minimal covering set (MC)
- Bipartisan set (BP)
- Bank’s solution (B)
- Tournament equilibrium set (TEQ)

Method for ranking based on the notion of covering Game theory based Based on Contestation

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Recall: Copeland score $s(x) = |T^+(x)| = |\{y \in X \mid xTy\}|$
s(x) is the number of alternatives that $x$ beats.

**Definition (Copeland solution (C))**

Copeland winners of $T \in \mathcal{F}(X)$ is
$C(T) = \{x \in X \mid \forall y \in X, s(y) = s(x)\}$

**Definition (Slater, Kandall, or Hamming distance)**

Let $(T, T') \in \mathcal{F}(X)$

$$\Delta(T, T') = \frac{1}{2} \# \{ (x, y) \in X^2 \mid xTy \land yT'x \}$$

How many arrows are flipped in the tournament graph?

**Definition (Slater order)**

Let $T \in \mathcal{F}(X)$.

A Slater order for $T$ is a linear order $U \in \mathcal{L}(X)$ such that

$$\Delta(T, U) = \min_{V \in \mathcal{L}(X)} \{ \Delta(T, V) \}$$

where $\mathcal{L}(X)$ is the set of linear order over $X$.

The set of Slater winners of $T$, noted $SL(T)$, is the set of alternatives in $X$ that are Condorcet winner of a Slater order for $T$.

idea: approximate the tournament by a linear order.
to make \( b, c, d \) a Condorcet winner, it needs “3 flips”
to make \( e \) a Condorcet winner, it needs “4 flips”

### Outline

1. Introduction: Reasoning about pairwise competition
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### Theorem

Computing a Slater ranking is \( \mathcal{NP} \)-hard.

- Vincent Conitzer, Computing Slater Rankings using similarities among candidates, AAAI, 2006

### Definition (Covering)

Let \( T \in \mathcal{P}(X) \) and \((x, y) \in X^2\)

\( x \) covers \( y \) in \( X \) if \( [xTy \text{ and } (\forall z \in X, yTz \Rightarrow xTz)] \)

We note \( x \triangleright y \)

### Definition (Equivalent definition of covering)

\( x \triangleright y \) if \( xTy \) and \( \forall z \in X, T^+(y) \subset T^+(x) \)

\( x \triangleright y \) if \( x \neq y \) and \( T^+(y) \subset T^+(x) \)

\( x \triangleright y \) if \( x \neq y \) and \( T^-(x) \subset T^-(y) \)
Definition (Uncovered Set (UC))

The uncovered set of \( T \) is \( UC(T) = \{ x \in X \mid \nexists y \in X \mid y \succ x \} \)


Any outcome \( x \) in the Uncovered Set either beats \( y \), or beats some \( z \) that beats \( y \) (\( x \) beats any other outcome it at most two steps).

Proposition

\[ \forall x \in X \setminus UC(X), UC^\infty(X) = UC^\infty(X \setminus \{x\}) \]

Find a covered alternative, remove it, continue...

Definition (Covering set)

Let \( T \in \mathcal{P}(X) \) and \( Y \subset X \).

\( Y \) is a **Covering set** for \( T \) if \( \forall x \in X \setminus Y, x \notin UC(Y \cup \{x\}) \).

(\( x \) is covered by some elements in \( Y \))

\( C(T) \) is the family of covering sets for \( T \).

Proposition

\[ \forall k \in (\mathbb{N} \cup \infty), UC^k(T) \text{ is a covering set for } T. \]

proposition

The family \( C(T) \) admits a minimal element (by inclusion) called the minimal covering set of \( T \) and denoted by \( MC(T) \).

$MC \subset UC^\infty$ and $MC \neq UC^\infty$

$UC(T) = X = UC^\infty(T)$

$MC(T) = \{1, 2, 3\}$

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**Definition (tournament game)**

A **tournament game** is a finite symmetric two-player game $(X, g)$ such that, $\forall (x, y) \in X^2$

- $g(x, y) + g(y, x) = 0$ (zero-sum game)
- $x \neq y \Rightarrow g(x, y) \in \{-1, 1\}$

$T \in \mathcal{T}(X) \leftrightarrow$ tournament game $(X, g)$ with $\forall (x, y) \in X^2$, $xTy$ iff $g(x, y) = +1$

**Theorem**

A tournament game has a unique Nash equilibrium in mixed strategy, and this equilibrium is symmetric.

**Definition (Bipartisan Set)**

Let $T \in \mathcal{T}(X)$.

The **Bipartisan set** $BP(X)$ is the support of the unique mixed equilibrium of the tournament game associated with $T$.

**Propositions**

- $y$ is a Condorcet winner $\Rightarrow \forall x \in X$, $y$ is a best response to $x$.
- $y$ is not a Condorcet winner $\Rightarrow \exists x | xTy$, $x$ is a best response to $y$.
- $(x, y)$ is a pure Nash equilibrium iff $
\begin{cases}
    x = y \\
x \text{ is a Condorcet winner}
\end{cases}$
- $x$ dominates $y$ in $(X, g) \Leftrightarrow x$ covers $y$
  - $UC(T)$ is the set of undominated strategies
  - $UC^\infty(T)$ is the set of strategies not sequentially dominated.
Is $y$ a good outcome?

For a solution tournament $\mathcal{S}$ and $T \in \mathcal{T}(X)$,
\[
\forall (x, y) \in X^2 \; x D(\mathcal{S}, T)y \Leftrightarrow x \in S(T | T^-(y))
\]
x is a contestation of $y$ for $T$ according to $\mathcal{S}$.

**Bank’s set**
There exists a unique tournament solution $B$ such that
\[
\forall T \in \mathcal{T}(X), \; o(T) \geq 2 \Rightarrow B(T) = D(B, T)^-(X)
\]
$D(B, T)^-(X)$ is the set of points in $X$ which are contestation of some point of $X$ according to $\mathcal{S}$.

**Proposition**
$x \in B(T)$ iff $\exists Y \subset X$ such that $x \in Y$ and $T|Y$ is an ordering for which $x$ is the winner and no point of $X$ beats all the points of $Y$. 

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\[ a \ Y = \{d\}, \ a \succ d \ \text{and} \ aTb, \ dTc, \ aTe. \quad \checkmark \]
\[ b \ Y = \{d, c\}, \ b \succ d \succ c \ \text{and} \ cTa, \ cTe. \quad \checkmark \]
\[ c \ Y = \{a\}, \ c \succ a \ \text{and} \ aTb, \ aTd, \ aTe. \quad \checkmark \]
\[ d \ Y = \{c, e\}, \ d \succ c \succ e \ \text{and} \ cTa, \ eTb. \quad \checkmark \]
\[ e \ Y = \{b\} \ \text{no because of} \ aTb \ \text{and} \ aTe. \]
\[ Y = \{b, c\} \ \text{not an ordering.} \quad \times \]
\[ B(T) = \{a, b, c, d\} \]

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**Definition (Algebraic solution)**

A tournament solution \( \mathcal{S} \) is **computable by a binary tree** if, for any order \( n \), there exists a labelled binary tree \((N, A, i)\) of order \( n \) such that, for any tournament \( T \in \mathcal{T}(X) \) of order \( n \), \( \mathcal{S}(T) \) is the set of winners of \( T \) along \((N, T, i)\) for all drawing of \( X \).

\( \mathcal{S} \) is computable by a binary tree iff \( \mathcal{S} \) is **algebraic**.

- Any algebraic tournament solution selects a winner in the top cycle.
- The Copeland and Markov solutions are not algebraic.
- Strengthening a winner can make her lose.
- There exists a non monotonous algebraic tournament solution.


Multistage elimination tree or sophisticated agenda

\[ \Gamma_n(1, 2, \ldots, n) = \Gamma_{n-1}(1, 2, \ldots, n-1) \]

\[ \Gamma_{n-1}(1, 2, \ldots, n-1) = \Gamma_{n-2}(1, 2, \ldots, n-2) \cdot \Gamma_n(1, 2, \ldots, n) \]

\[ \Gamma_2(1, 2) \]

\[ \Gamma_2(1, 2, 3) \]

\[ \Gamma_2(1, 2, 3, 4) \]


**Sophisticated voting on simple agendas**

- \( \Gamma_k(a) \): outcome of strategic voting on the simple agenda of order \( k \) with agenda \( a \)
- \( a_{-n} = a(1) \cdot a(2) \ldots a(n-2) \cdot a(n-1) \)
- \( a_{-(n-1)} = a(1) \cdot a(2) \ldots a(n-2) \cdot a(n) \ldots a(n) \)

Voting for \( a(n) \) or \( a(n-1) \) ⇒ Comparing \( \Gamma_{n-1}(a_{-n}) \) and \( \Gamma_{n-1}(a_{-(n-1)}) \), i.e.,

\[ \Gamma_n(a) = \Gamma_{n-1}(a_{-n}) \cdot \Gamma_{n-1}(a_{-(n-1)}) \]

**Sophisticated agenda and sophisticated voting**

Strategic voting on a simple agenda results in choosing the winner of the associated sophisticated agenda.

**Knockout tournaments**

**Definition (General Knockout Tournament)**

Given a set \( N \) of players and a matrix \( P \) such that \( P_{ij} \) denotes the probability that player \( i \) wins against player \( j \) in a pairwise elimination match and \( \forall (i, j) \in N^2 \ 0 \leq P_{ij} = 1 - P_{ji} \leq 1 \),

a knockout tournament \( KTN = (T, S) \) is defined by:

- A tournament structure \( T \): a binary tree with \( |N| \) leaf nodes
- A seeding \( S \): a bijection between the players in \( N \) and the leaf nodes of \( T \)

**Theorem**

It is \( \text{NP} \)-complete to decide whether there exists a tournament structure \( KTN \) with round placement \( R \) such that a target player \( k \in N \) will win the tournament.

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Properties

For Bipartisan set, minimal covering set, iterated uncovered set and the top cycle
- if ∃ a Condorcet winner, the winner is unique (definition)
- if ∄ a Condorcet winner, the set of winners contains at least 3 alternatives.

For all tournaments are equiprobable, the top cycle is almost surely the whole set of alternatives.
Probability that every alternative is in the Banks set in a random tournament goes to one as the number of alternatives goes to infinity. (every alternative is in the Banks set in almost all tournaments).


Bibliography