Introduction to Tournaments

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ILLC

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• **Voting**
  
  **Input:** Preference of agents over a set of candidates or outcomes  
  **Output:** one candidate or outcome (or a set)

• **Tournament**
  
  **Input:** Binary relation between outcomes or candidates  
  **Output:** One candidate or outcome (or a set)

When no ties are allowed between any two alternatives.  
Either $x$ beats $y$ or $y$ beats $x$.  

which are the best outcomes?
Notations

- $X$ is a finite set of alternatives.
- $T$ is a relation on $X$, i.e., $T \subset X^2$.
- Notation: $(x, y) \in T \iff xTy \iff x \rightarrow y \iff x$ “beats” $y$
- $\mathcal{T}(X)$ is the set of tournaments on $X$
- $T^+(x) = \{ y \in X \mid xTy \}$: successors of $x$.
- $T^-(x) = \{ y \in X \mid yTx \}$: predecessors of $x$.
- $s(x) = \#T^+(x)$ is the Copeland score of $x$. 

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Definition (Tournament)

The relation $T$ is a tournament iff

1. $\forall x \in X \ (x, x) \notin T$
2. $\forall (x, y) \in X^2 \ x \neq y \Rightarrow [(x, y) \in T) \lor ((y, x) \in T)]$
3. $\forall (x, y) \in X^2 \ (x, y) \in T \Rightarrow (y, x) \notin T$.

A tournament is a complete and asymmetric binary relation.

Majority voting and tournament:

- $I$ finite set of individuals. The preference of an individual $i$ is represented by a complete order $P_i$ defined on $X$.
- The outcome of majority voting is the binary relation $M(P)$ on $X$ such that $\forall (x, y) \in X, xM(P)y \iff \#\{i \in I \mid xP_i y\} > \#\{i \in I \mid yP_i x\}$

If initial preferences are strict and number of individual is odd, $M(P)$ is a tournament.
Example (cyclone of order $n$)

$Z_n$ set of integers modulo $n$.

$xC_ny \iff y - x \in \{1, \ldots, \frac{n-1}{2}\}$

$T^+(1) = \{2, 3, 4\}$

$T^-(1) = \{5, 6, 7\}$
Definition (isomorphism)

Let $X$ and $Y$ be two sets, $T \in \mathcal{T}(X)$, $U \in \mathcal{T}(Y)$ two tournaments on $X$ and $Y$.
A mapping $\phi : X \rightarrow Y$ is a tournament isomorphism iff
- $\phi$ is a bijection
- $\forall (x, y) \in X^2, xT x' \iff \phi(x) U \phi(x')$

On a set $X$ of cardinal $n$, there are $2^{\frac{n\cdot(n-1)}{2}}$ tournaments, but many of them are isomorphic.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$2^{\frac{n\cdot(n-1)}{2}}$</th>
<th>number of non-isomorphic tournaments</th>
</tr>
</thead>
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<tr>
<td>8</td>
<td>268,435,456</td>
<td>6,880</td>
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<tr>
<td>10</td>
<td>35,184,372,088,832</td>
<td>9,733,056</td>
</tr>
</tbody>
</table>
Outline

1. Introduction: Reasoning about pairwise competition
2. Desirable properties of solution concepts
3. Solution based on scoring and Ranking
4. Solutions based on Covering
5. Solution based on Game Theory
6. Contestation Process
7. Knockout tournaments
8. Notes on the size of the choice set
Definition (Condorcet winners)

Let $T \in \mathcal{T}(X)$. The set of Condorcet winners of $T$ is

$$\text{Condorcet}(T) = \{x \in X \mid \forall y \in X, y \neq x \Rightarrow xTy\}$$

Property

Either $\text{Condorcet}(T) = \emptyset$ or $\text{Condorcet}(T)$ is a singleton.
Definition (Tournament solution)

A tournament solution $\mathcal{S}$ associates to any tournament $\mathcal{T}(X)$ a subset $\mathcal{S}(T) \subset X$ and satisfies

- $\forall T \in \mathcal{T}(X), \mathcal{S}(T) \neq \emptyset$
- For any tournament isomorphism $\phi$, $\phi \circ \mathcal{S} = \mathcal{S} \circ \phi$ (anonymity)
- $\forall T \in \mathcal{T}(X), \text{Condorcet}(T) \neq \emptyset \Rightarrow \mathcal{S}(T) = \text{Condorcet}(T)$

For $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_2$ tournament solutions.

- $\mathcal{S}_1 \circ \mathcal{S}_2 (T) = \mathcal{S}_1 (T/\mathcal{S}_2 (T)) = \mathcal{S}_1 (\mathcal{S}_2 (T))$
- $\mathcal{S}^1 = \mathcal{S}$, $\mathcal{S}^{k+1} = \mathcal{S} \circ \mathcal{S}^k$, $\mathcal{S}^\infty = \lim_{k \to \infty} \mathcal{S}^k$

- solutions may be finer/more selective:
  \[ \mathcal{S}_1 \subset \mathcal{S}_2 \Leftrightarrow \forall T \in \mathcal{T}(X) \, \mathcal{S}_1 (T) \subset \mathcal{S}_2 (T) \text{ than } \mathcal{S}_2. \]

- solutions may be different:
  \[ \mathcal{S}_1 \varnothing \mathcal{S}_2 \Leftrightarrow \exists T \in \mathcal{T} \, | \, \mathcal{S}_1 (T) \cap \mathcal{S}_2 (T) = \emptyset \]

- solution may have common elements:
  \[ \mathcal{S}_1 \cap \mathcal{S}_2 \Leftrightarrow \forall T \in \mathcal{T} \, | \, \mathcal{S}_1 (T) \cap \mathcal{S}_2 (T) \neq \emptyset \]
A first solution: the Top Cycle (TC)

**Definition (Top Cycle)**

The top cycle of $T \in \mathcal{F}(X)$ is the set $TC$ defined as

$$TC(T) = \left\{ x \in X \mid \forall y \in X, \exists k > 0 \left( \exists (z_1, \ldots, z_k) \in X^k, z_1 = x, z_k = y, \text{and} 1 \leq i < j \leq k \implies z_i T z_j \right) \right\}$$

The top cycle contains outcomes that beat directly or indirectly every other outcomes.
Properties of Solutions

- Regular
- Monotonous
- Independent of the losers
- Strong Superset Property
- Idempotent
- Aïzerman property
- Composition-consistent and weak composition-consistent
Definition (Regular tournament)
A tournament is regular iff all the points have the same Copeland score.

Definition (Monotonous)
A solution $\mathcal{S}$ is monotonous iff $\forall T \in \mathcal{T}(X), \forall x \in \mathcal{S}(T), \forall T' \in \mathcal{T}(X)$ such that
\[
\begin{align*}
T'/X \setminus \{x\} &= T/X \setminus \{x\} \\
\forall y \in X, xTY &\Rightarrow xT'y
\end{align*}
\]
one has $x \in \mathcal{S}(T')$

“Whenever a winner is reinforced, it does not become a loser.”
Definition (Independence of the losers)

A solution $\mathcal{I}$ is independent of the losers iff $\forall T \in \mathcal{T}(X), \forall T' \in \mathcal{T}(X)$ such that $\forall x \in \mathcal{I}(T), \forall y \in X$, $xTy \iff xT'y$ one has $\mathcal{I}(T) = \mathcal{I}(T')$.

“the only important relations are \{ winners to winners \}

“What happens between losers do not matter.”

"the only important relations are \{ winners to winners, winners to losers \"

Definition (Strong Superset Property (SSP))

A solution $\mathcal{I}$ satisfies the Strong Superset Property (SSP) iff $\forall T \in \mathcal{T}(X), \forall Y \mid \mathcal{I}(T) \subset Y \subset X$ one has $\mathcal{I}(T) = \mathcal{I}(T/Y)$

“We can delete some or all losers, and the set of winners does not change”
Definition (Idempotent)

A solution $\mathcal{I}$ is idempotent iff $\mathcal{I} \circ \mathcal{I} = \mathcal{I}$.

Definition (Aïzerman property)

A solution $\mathcal{I}$ satisfies the Aïzerman property iff $\forall T \in \mathcal{T}(X), \forall Y \subset X$ $\mathcal{I}(T) \subset Y \subset X \Rightarrow \mathcal{I}(T/Y) \subset \mathcal{I}(T)$
### Solution Concepts

- **Copeland solution (C)**
- **the Long Path (LP)**
- **Markov solution (MA)**
- **Slater solution (SL)**
- **Uncovered set (UC)**
- **Iterations of the Uncovered set** $(UC^\infty)$
- **Dutta’s minimal covering set (MC)**
- **Bipartisan set (BP)**
- **Bank’s solution (B)**
- **Tournament equilibrium set (TEQ)**

- method for ranking
- based on the notion of covering
- Game theory based
- Based on Contestation
<table>
<thead>
<tr>
<th>Property</th>
<th>TC</th>
<th>UC</th>
<th>$UC^\infty$</th>
<th>MC</th>
<th>BP</th>
<th>B</th>
<th>TEQ</th>
<th>SL</th>
<th>C</th>
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<td>≤1/3</td>
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<td>$O(n^{2.38})$</td>
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\[\exists T \in T_{29} \mid B(T) \subset BP(T) \text{ and } B(T) \neq BP(T)\]
\[\exists T' \in T_6 \mid BP(T') \subset B(T') \text{ and } B(T') \neq BP(T').\]
It is unknown if \(B \cap BP\) can be empty.
Same for TEQ and BP.

b) \(TEQ \subset MC\) is a conjecture
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Recall: Copeland score $s(x) = |T^+(x)| = |\{y \in X \mid xTy\}|$

$s(x)$ is the number of alternatives that $x$ beats.

**Definition (Copeland solution (C))**

Copeland winners of $T \in \mathcal{T}(X)$ is

$C(T) = \{x \in X \mid \forall y \in X, s(y) = s(x)\}$
Definition (Slater, Kandall, or Hamming distance)

Let \((T, T') \in \mathcal{T}(X)\)

\[
\Delta(T, T') = \frac{1}{2} \# \{(x, y) \in X^2 \mid xTy \land yT'x\}
\]

How many arrows are flipped in the tournament graph?

Definition (Slater order)

Let \(T \in \mathcal{T}(X)\).

A Slater order for \(T\) is a linear order \(U \in \mathcal{L}(X)\) such that

\[
\Delta(T, U) = \min_{V \in \mathcal{L}(X)} \{\Delta(T, V)\}
\]

where \(\mathcal{L}(X)\) is the set of linear order over \(X\).

The set of Slater winners of \(T\), noted \(SL(T)\), is the set of alternatives in \(X\) that are Condorcet winner of a Slater order for \(T\).

idea: approximate the tournament by a linear order.
to make $b$, $c$, $d$ a Condorcet winner, it needs “3 flips”
to make $e$ a Condorcet winner, it needs “4 flips”
Theorem

Computing a Slater ranking is \( \mathcal{NP} \)-hard.


Vincent Conitzer, Computing Slater Rankings using similarities among candidates, AAAI, 2006
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**Definition (Covering)**

Let \( T \in \mathcal{T}(X) \) and \((x, y) \in X^2\)

\( x \) covers \( y \) in \( X \) iff \([xTy \text{ and } (\forall z \in X, yTz \Rightarrow xTz)]\)

We note \( x \succ y \)

---

**Definition (Equivalent definition of covering)**

- \( x \succ y \) iff \( xTy \) and \( \forall z \in X, \frac{T}{\{x,y,z\}} \) is transitive.
- \( x \succ y \) iff \( x \neq y \) and \( T^+(y) \subset T^+(x) \)
- \( x \succ y \) iff \( x \neq y \) and \( T^-(x) \subset T^-(y) \)
Definition (Uncovered Set (UC))

The uncovered set of $T$ is $UC(T) = \{ x \in X \mid \exists y \in X \mid y \succ x \}$


Any outcome $x$ in the Uncovered Set either beats $y$, or beats some $z$ that beats $y$ ($x$ beats any other outcome it at most two steps).
A tournament

\[ UC(T) = \{ a, b, c, d \} \]
Proposition

$$\forall x \in X \setminus UC(X), \ UC^\infty(X) = UC^\infty(X \setminus \{x\})$$

Find a covered alternative, remove it, continue...

$$T \setminus \{a, b, c, d\}$$

$$UC(T \setminus \{a, b, c, d\}) = \{a, b, c\}$$

$$T \setminus \{a, b, c\}$$

covering relation ▷

covering relation ▷
Definition (Covering set)

Let \( T \in \mathcal{P}(X) \) and \( Y \subset X \).

\( Y \) is a **Covering set** for \( T \) iff \( \forall x \in X \setminus Y, \ x \notin UC(Y \cup \{x\}) \).

\( x \) is covered by some elements in \( Y \)

\( C(T) \) is the family of covering sets for \( T \).

Proposition

\( \forall k \in (\mathbb{N} \cup \infty), \ UC^k(T) \) is a covering set for \( T \).

Proposition

The family \( C(T) \) admits a minimal element (by inclusion) called the **minimal covering set** of \( T \) and denoted by \( MC(T) \).

\[ MC \subset UC^\infty \text{ and } MC \neq UC^\infty \]

\[ UC(T) = X = UC^\infty(T) \]

\[ MC(T) = \{1, 2, 3\} \]
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Definition (tournament game)

A tournament game is a finite symmetric two-player game \((X, g)\) such that, \(\forall (x, y) \in X^2\)

- \(g(x, y) + g(y, x) = 0\) (zero-sum game)
- \(x \neq y \Rightarrow g(x, y) \in \{-1, 1\}\)

\(T \in \mathcal{T}(X) \iff\) tournament game \((X, g)\)
with \(\forall (x, y) \in X^2, xTy \iff g(x, y) = +1\)

Propositions

- \(y\) is a Condorcet winner \(\Rightarrow \forall x \in X, y\) is a best response to \(x\).
- \(y\) is not a Condorcet winner \(\Rightarrow \forall x | xTy, x\) is a best response to \(y\).
- \((x, y)\) is a pure Nash equilibrium iff \(\begin{cases} x = y \\ x\ is\ a\ Condorcet\ winner \end{cases}\)
- \(x\) dominates \(y\) in \((X, g) \iff x\ covers\ y\)
  - \(UC(T)\) is the set of undominated strategies
  - \(UC^\infty(T)\) is the set of strategies not sequentially dominated.
Theorem

A tournament game has a unique Nash equilibrium in mixed strategy, and this equilibrium is symmetric.

Definition (Bipartisan Set)

Let $T \in \mathcal{T}(X)$. The Bipartisan set $BP(X)$ is the support of the unique mixed equilibrium of the tournament game associated with $T$. 
\begin{array}{c|ccccc}
\rightarrow & a & b & c & d & e \\
a & 0 & 1 & -1 & 1 & 1 \\
b & -1 & 0 & 1 & 1 & -1 \\
c & 1 & -1 & 0 & -1 & 1 \\
d & -1 & -1 & 1 & 0 & 1 \\
e & -1 & 1 & -1 & -1 & 0 \\
\end{array}
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Is $y$ a good outcome?

For a solution tournament $\mathcal{I}$ and $T \in \mathcal{T}(X)$,

$$\forall (x, y) \in X^2 \ x D(\mathcal{I}, T)y \Leftrightarrow x \in S(T | T^-(y))$$

$x$ is a contestation of $y$ for $T$ according to $\mathcal{I}$. 
Bank’s set

There exists a unique tournament solution $B$ such that

$$\forall T \in \mathcal{T}(X), \ o(T) \geq 2 \ \Rightarrow \ B(T) = D(B, T)^-(X)$$

$D(B, T)^-(X)$ is the set of points in $X$ which are contestation of some point of $X$ according to $\mathcal{T}$.

Proposition

$x \in B(T)$ iff $\exists Y \subset X$ such that $x \in Y$ and $T|Y$ i an ordering for which $x$ is the winner and no point of $X$ beats all the points of $Y$. 
\begin{itemize}
  \item[a] \( Y = \{d\} \), \( a \succ d \) and \( aTb, \, dTc, \, aTe \). \( \checkmark \)
  \item[b] \( Y = \{d, c\} \), \( b \succ d \succ c \) and \( cTa, \, cTe \). \( \checkmark \)
  \item[c] \( Y = \{a\} \), \( c \succ a \) and \( aTb, \, aTd, \, aTe \). \( \checkmark \)
  \item[d] \( Y = \{c, e\} \), \( d \succ c \succ e \) and \( cTa, \, eTb \). \( \checkmark \)
  \item[e] \( Y = \{b\} \) \textbf{no} because of \( aTb \) and \( aTe \).

  \( Y = \{b, c\} \) \textbf{not} an ordering. \( \times \)
\end{itemize}

\( B(T) = \{a, b, c, d\} \)
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Definition (Algebraic solution)

A tournament solution $\mathcal{S}$ is **computable by a binary tree** if, for any order $n$, there exists a labelled binary tree $(N, A, i)$ of order $n$ such that, for any tournament $T \in \mathcal{T}(X)$ of order $n$, $\mathcal{S}(T)$ is the set of winners of $T$ along $(N, T, i)$ for all drawing of $X$.

$\mathcal{S}$ is computable by a binary tree iff $\mathcal{S}$ is **algebraic**.

- Any algebraic tournament solution selects a winner in the top cycle.
- The Copeland and Markov solutions are not algebraic.
- Strengthening a winner can make her lose.
- There exists a non monotonous algebraic tournament solution.


Multistage elimination tree or sophisticated agenda

\[ \Gamma_{n-1}(1, 2, \ldots, n-1, n) \quad \Gamma_{n-1}(1, 2, \ldots, n-1) \]

\[ \Gamma_n(1, 2, \ldots, n) \]

\[ \Gamma_2(1, 2) \quad \Gamma_2(1, 2, 3) \quad \Gamma_2(1, 2, 3, 4) \]


Sophisticated voting on simple agendas

- $\Gamma_k(a)$: outcome of *strategic* voting on the simple agenda of order $k$ with agenda $a$
- $a_{-n} = a(1) \cdot a(2) \ldots a(n-2) \cdot a(n-1)$
- $a_{-(n-1)} = a(1) \cdot a(2) \ldots a(n-2) \cdot a(n) \ldots a(n)$

Voting for $a(n)$ or $a(n-1)$ $\Rightarrow$ Comparing $\Gamma_{n-1}(a_{-n})$ and $\Gamma_{n-1}(a_{-(n-1)})$, i.e.,

$$\Gamma_n(a) = \Gamma_{n-1}(a_{-n}) \cdot \Gamma_{n-1}(a_{-(n-1)})$$

Sophisticated agenda and sophisticated voting

Strategic voting one a simple agenda results in choosing the winner of the associated sophisticated agenda.
Property

Let $\mathcal{B}$ the set of all permutations of $X = \{1, \ldots, n\}$
Let $a \in \mathcal{B}$, $w(\Gamma_n, T, a)$ is the winner of the tournament $T \in \mathcal{T}(X)$ along the sophisticated agenda $\Gamma_n$ for the drawing $a$.

$$\{w(\Gamma_n, T, a), a \in \mathcal{B}\} = Bank(T)$$
Knockout tournaments

Definition (General Knockout Tournament)

Given a set $N$ of players and a matrix $P$ such that $P_{ij}$ denotes the probability that player $i$ wins against player $j$ in a pairwise elimination match and $\forall (i, j) \in N^2$ $0 \leq P_{ij} = 1 - P_{ji} \leq 1$, a knockout tournament $KTN = (T, S)$ is defined by:

- A tournament structure $T$: a binary tree with $|N|$ leaf nodes
- A seeding $S$: a bijection between the players in $N$ and the leaf nodes of $T$

Theorem

It is $\mathcal{NP}$-complete to decide whether there exists a tournament structure $KT$ with round placement $R$ such that a target player $k \in N$ will win the tournament.

1. Introduction: Reasoning about pairwise competition
2. Desirable properties of solution concepts
3. Solution based on scoring and Ranking
4. Solutions based on Covering
5. Solution based on Game Theory
6. Contestation Process
7. Knockout tournaments
8. Notes on the size of the choice set
Properties

For Bipartisan set, minimal covering set, iterated uncovered set and the top cycle

- if $\exists$ a Condorcet winner, the winner is unique (definition)
- if $\nexists$ a Condorcet winner, the set of winners contains at least 3 alternatives.

Properties

If all tournaments are equiprobable, the top cycle is almost surely the whole set of alternatives.
Probability that every alternative is in the Banks set in a random tournament goes to one as the number of alternatives goes to infinity. (*every* alternative is in the Banks set in *almost all* tournaments).

