Plan for Today

We have seen already that we need to be precise about the properties we would like to see in a voting procedure and that it can be hard to satisfy all the desiderata we might have. Using the axiomatic method, today we will see two impossibility theorems:

- Arrow’s Theorem [1951]
- the Muller-Satterthwaite Theorem [1977]

This is (very) classical social choice theory, but we will also briefly touch upon some modern COMSOC concerns:

- Can we go beyond the mathematical rigour of SCT and achieve a formalisation in the sense of symbolic logic?
- Can we automate the proving of theorems in SCT?
- What changes if we alter the notion of ballot, which classically is assumed to be a (usually strict) ranking of the alternatives?

Formal Framework

Basic terminology and notation:

- finite set of voters $\mathcal{N} = \{1, \ldots, n\}$, the electorate
- (usually finite) set of alternatives $\mathcal{X} = \{x_1, x_2, x_3, \ldots\}$
- Denote the set of linear orders on $\mathcal{X}$ by $\mathcal{L}(\mathcal{X})$. Preferences are assumed to be elements of $\mathcal{L}(\mathcal{X})$. Ballots are elements of $\mathcal{L}(\mathcal{X})$.

A voting procedure is a function $F : \mathcal{L}(\mathcal{X})^\mathcal{N} \to 2^\mathcal{X} \setminus \{\emptyset\}$, mapping profiles of ballots to nonempty sets of alternatives.

Remark 1: Approval Voting, Majority Judgment, Cumulative and Range Voting don’t fit this framework; everything else we’ve seen does.

Remark 2: If we wanted to be a bit more general, we could introduce a ballot language $\mathcal{B}(\mathcal{X})$ and work with functions $F : \mathcal{B}(\mathcal{X})^\mathcal{N} \to 2^\mathcal{X} \setminus \{\emptyset\}$.

Remark 3: A voting procedure parametrised by $\mathcal{N}$ and $\mathcal{X}$ (e.g., Borda) is a family of functions $F^{\mathcal{N},\mathcal{X}} : \mathcal{L}(\mathcal{X})^\mathcal{N} \to 2^\mathcal{X} \setminus \{\emptyset\}$.

Resoluteness and Tie-Breaking

$F : \mathcal{L}(\mathcal{X})^\mathcal{N} \to 2^\mathcal{X} \setminus \{\emptyset\}$ is called resolute if $|F(b)| = 1$ for any ballot profile $b \in \mathcal{L}(\mathcal{X})^\mathcal{N}$, i.e., if $F$ always produces a unique winner.

Terminology: voting rule vs. voting correspondence
(resolute) (irresolute)

We can turn an irresolute procedure $F$ into a resolute procedure $F \circ T$ by pairing $F$ with a (deterministic) tie-breaking rule $T : 2^\mathcal{X} \setminus \{\emptyset\} \to \mathcal{X}$ with $T(X) \in X$ for any $X \in 2^\mathcal{X} \setminus \{\emptyset\}$. Examples:

- select the lexicographically first alternative
- select the preferred alternative of some chair person

We will (mostly) just analyse either irresolute or resolute procedures, without worrying about tie-breaking in particular.
The Axiomatic Method

Many important classical results in social choice theory are axiomatic. They formalise desirable properties as "axioms" and then establish:

- Characterisation Theorems, showing that a particular (class of) procedure(s) is the only one satisfying a given set of axioms
- Impossibility Theorems, showing that there exists no voting procedure satisfying a given set of axioms

Today, we will see two examples for the latter. We first discuss some of these axioms, starting with very basic ones.

Universal Domain

The first axiom is not really an axiom . . . Sometimes the fact that voting procedures $F$ are defined over all ballot profiles is stated explicitly as a universal domain axiom.

Instead, I prefer to think of this as an integral part of the definition of the framework (for now) or as a domain condition (later on).

Anonymity and Neutrality

A voting rule is anonymous if the voters are treated symmetrically: if two voters switch ballots, then the winners don’t change. Formally:

$F$ is anonymous if $F(b_1, \ldots, b_n) = F(b_{\pi(1)}, \ldots, b_{\pi(n)})$ for any ballot profile $(b_1, \ldots, b_n)$ and any permutation $\pi : N \rightarrow N$.

A voting procedure is neutral if the alternatives are treated symmetrically. Formally:

$F$ is neutral if $F(\pi(b)) = \pi(F(b))$ for any ballot profile $b$ and any permutation $\pi : \mathcal{X} \rightarrow \mathcal{X}$ (with $\pi$ extended to ballot profiles and sets of alternatives in the natural manner).

Nonimposition

A voting procedure satisfies nonimposition if each alternative is the unique winner under at least one ballot profile. Formally:

$F$ satisfies nonimposition if for any alternative $x \in \mathcal{X}$ there exists a ballot profile $b \in \mathcal{L}(\mathcal{X})^N$ such that $F(b) = \{x\}$.

Remark 1: Any surjective (onto) voting procedure satisfies nonimposition. For resolute procedures, the two properties coincide.

Remark 2: Any neutral resolute voting procedure satisfies nonimposition.
Dictatorships

A voting procedure is **dictatorial** if there exists a voter (the dictator) such that the unique winner will always be her top-ranked alternative.

A voting procedure is **nondictatorial** if it is not dictatorial. Formally:

\[ F \text{ is nondictatorial if there exists no voter } i \in N \text{ such that } F(b) = \{ x \} \text{ whenever } i \in b(x \succ y) \text{ for all } y \in X \setminus \{ x \}. \]

**Remark:** Any **anonymous** voting procedure is nondictatorial.

Notation: \( b(x \succ y) \) is the set of voters ranking \( x \) above \( y \) in profile \( b \).

Unanimity and the Pareto Condition

A voting procedure is **unanimous** if it elects (only) \( x \) whenever all voters say that \( x \) is the best alternative. Formally:

\[ F \text{ is unanimous if } b(x \succ y) = N \text{ for all } y \in N \setminus \{ x \} \text{ implies } F(b) = \{ x \}. \]

The **weak Pareto condition** holds if an alternative \( y \) that is dominated by some other alternative \( x \) in all ballots cannot win. Formally:

\[ F \text{ is weakly Pareto if } b(x \succ y) = N \text{ implies } y \notin F(b). \]

**Remark:** The weak Pareto condition entails unanimity, but the converse is not true.

Independence of Irrelevant Alternatives (IIA)

A voting procedure is **independent of irrelevant alternatives (IIA)** if, whenever \( y \) loses to some winner \( x \) and the relative ranking of \( x \) and \( y \) does not change in the ballots, then \( y \) cannot win (independently of any possible changes wrt. other, irrelevant, alternatives). Formally:

\[ F \text{ satisfies IIA if } x \in F(b) \text{ and } y \notin F(b) \text{ together with } b(x \succ y) = b'(x \succ y) \text{ imply } y \notin F(b') \text{ for any profiles } b \text{ and } b'. \]

**Remark:** IIA was introduced by Arrow (1951/1963), originally for **social welfare functions** (SWFs), where the outcome is a preference ordering. Above variant of IIA (for voting) is due to Taylor (2005).

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**Arrow’s Theorem for Voting Procedures**

This is widely regarded as the seminal result in social choice theory. Kenneth J. Arrow received the Nobel Prize in Economics in 1972.

**Theorem 1 (Arrow, 1951)** No voting procedure for \( \geq 3 \) alternatives can be weakly Pareto, IIA, and nondictatorial.

This particular version of the theorem is due to Taylor (2005).

Maybe the most accessible proof (of the standard formulation of the theorem) is the first proof in the paper by Geanakoplos (2005).

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Remarks

- Note that this is a surprising result!
- Note that the theorem does not hold for two alternatives.
- We can interpret the theorem as a characterisation result:
  A voting procedure for \( \geq 3 \) alternatives satisfies the weak Pareto condition and IIA if and only if it is a dictatorship.
- IIA is the most debatable of the three axioms featuring in the theorem. Indeed, it is quite hard to satisfy.
- The importance of Arrow’s Theorem is due to the result itself (“there is no good way to aggregate preferences!”), but also to the method: for the first time (a) the desiderata had been rigorously specified and (b) an argument was given that showed that there can be no good procedure (rather than just pointing out flaws in concrete existing procedures).

Proof of Arrow’s Theorem

We’ll sketch a proof adapted from Sen (1986), who proves the standard formulation of Arrow’s Theorem using the “decisive coalition” technique.

Definitions:

- A voting procedure for \( \geq 3 \) alternatives satisfies the weak Pareto condition if and only if it is a dictatorship.

Proof Plan:

- Pareto condition: \( \mathcal{N} \) is decisive for all pairs
- Lemma: \( G \) with \( |G| > 1 \) decisive for all pairs \( \Rightarrow \) some \( G' \subset G \) as well
- Thus (by induction), there’s a decisive coalition of size 1 (a dictator).

The proof of the lemma relies on another lemma:

- Lemma: \( G \) almost decisive for some \( (x, y) \) \( \Rightarrow G \) decisive for all \( (a, b) \)

Proof of the Decisiveness Lemma

Suppose \( F \) is a voting procedure that is weakly Pareto and IIA.

Lemma 1 (Decisiveness) If \( G \subseteq \mathcal{N} \) is an almost decisive coalition for some pair \( (x, y) \in \mathcal{X}^2 \), then \( G \) is decisive for all pairs \( (a, b) \in \mathcal{X}^2 \).

Proof: Suppose \( x, y, a, b \) are all distinct (other cases work similarly).

Consider this (class of) ballot profile(s):

\[
G: \quad a \succ x \succ y \succ b \\
Rest: \quad a \succ x \text{ and } y \succ b \text{ and } y \succ x \text{ (rest unspecified)}
\]

Note that “the rest” could have any ranking for \( a \) and \( b \).

From \( G \) being almost decisive on \((x, y)\) \( \Rightarrow \) \( y \) loses to \( x \)
From the weak Pareto condition \( \Rightarrow \) \( x \) loses to \( a \) and \( b \) loses to \( y \)

Thus, \( b \) loses and (only) \( a \) wins in a situation where \( G \) has \( a \succ b \), independently of how the rest rank \( a \) and \( b \). So, by IIA, \( b \) will lose to \( a \) for any profile in which \( G \) has \( a \succ b \), i.e., \( G \) is decisive for \((a, b)\). \( \checkmark \)

Proof of the Contraction Lemma

Lemma 2 (Contraction) If a coalition \( G \subseteq \mathcal{N} \) with \( |G| > 1 \) is decisive for all pairs, then so is some smaller coalition \( G' \subset G \).

Proof: Let \( G = G_1 \cup G_2 \), both nonempty. Consider this ballot profile:

\[
G_1: \quad x \succ y \succ z \quad \Rightarrow \text{ as } G \text{ decisive, } z \text{ loses (against } y) \\
G_2: \quad y \succ z \succ x \quad \text{thus, two possibilities:} \\
Rest: \quad z \succ x \succ y \quad \text{(1) } x \text{ wins or (2) only } y \text{ wins}
\]

Case (1): \( x \) wins, \( z \) loses, and only \( G_1 \) has \( x \succ z \)

- by IIA, \( z \) will lose for any profile in which only \( G_1 \) has \( x \succ z \)
- in other words, \( G_1 \) is almost decisive for \((x, z)\)
- by Lemma 1, \( G_1 \) is thus decisive for all pairs

Case (2): \( y \) wins, \( x \) loses, and only \( G_2 \) has \( y \succ x \)

- by the same argument, \( G_2 \) is decisive for all pairs

Hence, either \( G_1 \) or \( G_2 \) will be decisive under all circumstances. \( \checkmark \)
Logic and Automated Reasoning

Logic has long been used to formally specify computer systems, facilitating formal or even automatic verification of various properties. Can we apply this methodology also to social choice mechanisms?

- What logic fits best?
- Which automated reasoning methods are useful?

Related Work

- Ågotnes et al. (2010) propose a modal logic to model preferences and their aggregation that can express Arrow’s Theorem.
- Arrow’s Theorem holds iff the set $T_{Arrow}$ of FOL formulas (defined in the paper) has no finite models (Grandi and E., 2009).
- Nipkow (2009) formalises and verifies a known proof of Arrow’s Theorem in the HOL proof assistant Isabelle.


Computer-aided Proof of Arrow’s Theorem

Tang and Lin (2009) prove two inductive lemmas:

- If there exists an Arrovian aggregator for $n$ voters and $m+1$ alternatives, then there exists one for $n$ and $m$ (if $n \geq 2$, $m \geq 3$).
- If there exists an Arrovian aggregator for $n+1$ voters and $m$ alternatives, then there exists one for $n$ and $m$ (if $n \geq 2$, $m \geq 3$).

Tang and Lin then show that the “base case” of Arrow’s Theorem with 2 voters and 3 alternatives can be fully modelled in propositional logic.

A SAT solver can verify Arrow(2,3) to be correct in < 1 second — that’s $(3^3)^3 \times 3! \approx 10^{28}$ aggregators [SWFs] to check.

Discussion: Opens up opportunities for quick sanity checks of hypotheses regarding new impossibility theorems.


Monotonicity

Next we want to formalise the idea that when a winner receives increased support, she should not become a loser.

We restrict attention to resolute voting procedures (unique winner).

- **Weak monotonicity**: $F$ is weakly monotonic if $F(b) = \{x\}$ implies $F(b') = \{x\}$ for any alternative $x$ and any two ballot profiles $b$ and $b'$ with $b(x \succ y) \subseteq b'(x \succ y)$ and $b(y \succ z) = b'(y \succ z)$ for all alternatives $y$ and $z$ different from $x$.

- **Strong monotonicity**: $F$ is strongly monotonic if $F(b) = \{x\}$ implies $F(b') = \{x\}$ for any alternative $x$ and any two ballot profiles $b$ and $b'$ with $b(x \succ y) \subseteq b'(x \succ y)$ for all alternatives $y$ different from $x$.

Strong monotonicity is also known as Maskin monotonicity or strong positive association.
Example

Even weak monotonicity is not satisfied by some voting procedures. Consider Plurality with Run-off (with some tie-breaking rule).

| 27 voters: | $A > B > C$ |
| 42 voters: | $C > A > B$ |
| 24 voters: | $B > C > A$ |

$B$ is eliminated in the first round and $C$ beats $A$ 66:27 in the run-off. But if 4 of the voters in the first group raise $C$ to the top (i.e., join the second group), then $B$ will win.

But other procedures (e.g., Plurality) do satisfy weak monotonicity. How about strong monotonicity?

The Muller-Satterthwaite Theorem

Strong monotonicity turns out to be (desirable but) too demanding:

**Theorem 2 (Muller and Satterthwaite, 1977)** No resolute voting procedure for $\geq 3$ alternatives can be surjective, strongly monotonic, and nondictatorial.

Proof omitted. The “decisive coalition” technique used to prove Arrow’s Theorem is applicable here as well (see e.g. Myerson, 1996).

Remark: Above theorem, which is what is nowadays usually referred to as the Muller-Satterthwaite Theorem, is in fact a corollary of their main theorem and the Gibbard-Satterthwaite Theorem.


Summary

This has been an introduction to the axiomatic method:

- formulate desirable properties of voting procedures as axioms
- explore the consequences of imposing several such axioms

We have seen two classical impossibility theorems that apply as soon as we have three or more alternatives:

- **Arrow’s Theorem**: only dictatorial rules are weakly Pareto and IIA
- **Muller-Satterthwaite Theorem**: only dictatorial rules are resolute, surjective, and strongly monotonic

Regarding modern COMSOC concerns we have discussed:

- using logic to fully formalise social choice problems and theorems, and automated reasoning to support analysis
- exploring ballot languages other than linear orders

Remark: Note that so far we have not made any (formal) use of the notion of “preference”; we have only talked about “ballots”. This will not change until we start discussing strategic issues.
What next?

As discussed, the impossibility theorems we have seen today could be interpreted as different axiomatic characterisations of the class of dictatorial voting procedures.

Next, we will see characterisations of more attractive (classes of) voting procedures:

- using (again) the axiomatic method; and
- using different methods.