Computational Social Choice - Autumn 2010
Judgment Aggregation

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Introduction

Voting theory deals with the problem of aggregating preferences:
From a set of weak or linear orders decide who is a/the winner.

Today we will study the problem of aggregating judgments, i.e., acceptance/rejection of several correlated propositions:

- Everything starts from the doctrinal paradox: majority voting over a simple set of correlated propositions leads to an inconsistent outcome
- This can be generalised defining a formal framework for judgment aggregation on propositional logic
- Representation, impossibility and possibility results can be proved, just like what you have seen in voting theory

In the second part we will see some COMSOC research topic in JA:

- Complexity of guaranteeing consistency of an aggregation procedure
- Define actual procedures and study complexity of standard problems like winner determination and strategic manipulation
- Explore the relation between judgment and preference aggregation

Part I:
An Introduction to Judgment Aggregation

Doctrinal Paradox - Discursive Dilemma

A story:
There is a court with three judges. Suppose legal doctrine stipulates that the defendant is liable if and only if there has been a valid contract (p) and that contract has been breached (q). The judgment is made by majority.

Doctrinal Paradox

	p	q	$p \wedge q$
Judge 1:	Yes	Yes	Yes
Judge 2:	No	Yes	No
Judge 3:	Yes	No	No
Majority:	Yes	Yes	No

Each individual judge is rational (i.e., has a consistent judgment)
but the majority is contradictory!

Kornhauser and Sager, Unpacking the court. Yale Law Journal, 1986.
Kornhauser and Sager, The one and the many: Adjudication... Calif. Law Review, 1993.
Precursors: Guilbaud (1966), Vacca (1922).

Basic Definitions I

JA was developed to generalise and study paradoxical situations that arise when a collective judgment has to be made on a set of correlated propositions

Ingredients:

- A finite set $\mathcal{N}$ of individuals
- A finite set Φ of propositional formulas called the agenda
- A judgment set is a subset of Φ indicating which formulas are accepted

If α is a propositional formula, define its complement $\sim \alpha$ as $\neg\alpha$ is α was not negated, otherwise β in case $\alpha = \neg\beta$.

Definition

An agenda is a finite subset of propositional formulas $\Phi \subseteq \mathcal{L}_{PS}$ closed under complementation and not containing double negations.

Basic Definitions II

A judgment set on an agenda Φ is a subset $J \subseteq \Phi$.

Call a judgment set J:

Complete: if for all $\alpha \in \Phi$ either α or its complement is in J.

Complement-free: if α and its complement $\sim \alpha$ are never both in J.

Consistent: there is an assignment to make all formulas in J true.

We assume that every individual submits a consistent and complete judgment set over the agenda (just in the same way as we assume linear orders for voting theory). Call $J(\Phi)$ the set of all consistent and complete judgment sets over Φ.

Definition

An aggregation procedure for agenda Φ and a set $\mathcal{N}$ of individuals is a function $F : J(\Phi)^{\mathcal{N}} \to 2^{\Phi}$, assigning a set of accepted propositions to every profile of consistent and complete judgment sets.

Axioms for Aggregation Procedures I

A first axiom regulates the properties of the output of aggregation:

Weak Rationality (WR): $F(\mathbf{J})$ is complete and complement-free.

Addendum (**Non-null**): if $\bot \in \Phi$ there exists a $\mathbf{J}$ such that $\bot \notin F(\mathbf{J})$.

Other standard requirements:

Unanimity (U): If $\varphi \in J_i$ for all i then $\varphi \in F(\mathbf{J})$.

Anonymity (A): F is symmetric with respect to individuals.

Non-dictatorship (ND): There exists no i such that $F(\mathbf{J}) = J_i$ for all $\mathbf{J}$.

Axioms for Aggregation Procedures II

The aggregation is not "alternative-dependent": if φ and ψ share the same pattern of individuals' judgments then their outcome must be the same:

> **Neutrality** (N):*For any φ, ψ in the agenda Φ and profile $\mathbf{J}$ in $J(\Phi)$, if $\varphi \in J_i \Leftrightarrow \psi \in J_i$ for all i, then $\varphi \in F(\mathbf{J}) \Leftrightarrow \psi \in F(\mathbf{J})$.*

The aggregation does not depends on the particular situation (profile): The outcome of F over φ depends solely on the individuals' judgments over φ:

> **Independence** (I): *For any φ in the agenda Φ and profiles $\mathbf{J}$ and $\mathbf{J}'$ in $J(\Phi)$, if $\varphi \in J_i \Leftrightarrow \varphi \in J'_i$ for all i, then $\varphi \in F(\mathbf{J}) \Leftrightarrow \varphi \in F(\mathbf{J}')$.*

Call systematic a function that is both independent and neutral. Define monotonicity in a similar way as in voting theory.

> **Systematicity** (S)=(N)+(I).
>
> **Monotonicity** (M): *Increased support for an accepted formula does not change its acceptance.*

We can play with this formalism to get (small) interesting results:

Lemma

If an agenda Φ contains a tautology, then every aggregation procedure for Φ that satisfies (WR), (N) *and* (I) *is unanimous* (U).

Proof.

If $\varphi^\top$ is a tautology then $\varphi^\top \in J_i$ for all $i \in \mathcal{N}$ (by individual rationality). By non-nullness there is a certain profile $\mathbf{J}$ such that $\varphi^\top \in F(\mathbf{J})$.
Consider now a formula ψ that is unanimously accepted in $\mathbf{J}'$: we have that $\psi \in J_i' \Leftrightarrow \varphi^\top \in J_i'$. Use (N) to deduce that the acceptance of φ must concord with that of $\varphi^\top$ in $\mathbf{J}'$, and use (I) to conclude that they both have to be accepted.

$\square$

An impossibility result without any logical consistency requirement.

Lemma

If the number of individuals is even, then there exists no aggregation procedure that satisfies (WR), (A), (N) *and* (I).

Proof.

By (N), (I) and (A) the acceptance of a formula φ depends only on the number of individuals supporting φ in profile $\mathbf{J}$. The profile where half of individuals accept φ and half accept $\neg\varphi$ is in contradiction with (WR). $\square$

Representation Result

Definition

Given an agenda Φ and an odd number of individuals, the majority rule accepts a formula φ if at least $\frac{n+1}{2}$ of the individuals accepts it.

Proposition

Given an agenda Φ and an odd number of individuals, the only aggregation procedure satisfying (WR), (A), (N), (I) *and* (M) *is the majority rule.*

Proof.
Believe me. $\square$

Impossibility Result

Call an agenda Φ rich if it contains at least two atoms p and q and their conjunction $p \wedge q$ (there are other equivalent definitions).

Theorem (List and Pettit)

Given a rich agenda Φ, there exists no consistent aggregation procedure that satisfies (WR), (A), (N) *and* (I).

Proof.

See blackboard (if there is time, otherwise see the paper). $\square$

List and Pettit, Aggregating sets of judgments: an Impossibility Result. Economics and Philosophy, 2002

Agenda Characterisation Result

Definition

An agenda Φ satisfies the median property iff every inconsistent subset of Φ contains an inconsistent subset of size at most 2.

Proposition

For more than 3 individuals, majority rule is consistent on an agenda Φ if and only if the Φ satisfies the median property.

Proof.

See blackboard. $\square$

Adapted from:

Nehring and Puppe, The structure of strategy-proof social choice... JET, 2007.

General Picture

- Plethora of impossibility theorems and agenda characterisation results
- Escapes from impossibility:
  - domain restrictions generalising single-peakedness
  - drop completeness of the output (see Adil's presentation)
  - define actual procedures: premise-based, distance-based procedures (see Part II)
- Strategy-proofness in JA (see Part II)
- Judgment Aggregation in more general logics

For a detailed introduction, see the following introductory paper:

List, Judgment Aggregation: A Short Introduction. Manuscript, LSE, 2008.

And the following (more technical) survey:

List and Puppe, Judgment Aggregation: A Survey. In P. Anand et al. (eds.), Handbook of Rational and Social Choice. Oxford University Press, 2009.

Part II:

Judgment aggregation at ILLC

Complexity of Judgment Aggregation

Classical problems:

Winner Determination - Strategy-proofness
Actual aggregation procedures have to be defined.
(wait a few slides)

New problem:

Consistency
Given an aggregation procedure over an agenda Φ,
is there a profile that generates an inconsistent outcome?

Connects to complexity of checking agenda properties (e.g. median property)

Safety of the Agenda

Axioms can be used to define different classes of aggregation procedures:

$$\begin{array}{c} \text{Set of axioms AX} \\ \text{Agenda } \Phi \end{array} \quad \Longrightarrow \quad \begin{array}{c} \text{Class of functions} \\ \mathcal{F}_{\Phi}[\text{AX}] \end{array}$$

Definition

An agenda Φ is safe with respect to a class of aggregation procedures $\mathcal{F}_{\Phi}$ if every function in $\mathcal{F}_{\Phi}$ is consistent.

This defines a complexity problem for every set AX of axioms: SAFETY[AX]

Endriss, Grandi and Porello, Complexity of Jugment Aggregation: Safety of the Agenda. Proceedings of AAMAS, 2010.

Complexity of Checking Safety (Independent Rules)

An agenda Φ satisfies the syntactic simplified median property (SSMP) if every nontrivially (i.e. not containing $\bot$) inconsistent subset of Φ has an inconsistent subset of the form $\{\varphi, \neg\varphi\}$.

Characterisation Result

Φ is safe for $\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{I}]$ if and only Φ satisfies the SSMP.

Theorem

$\mathrm{SAFETY}[\mathrm{WR}, \mathrm{A}, \mathrm{I}]$ *is Π_2^p-complete.*

Proof.

Φ is safe if and only if it satisfies the SSMP. Checking SSMP of an agenda is Π_2^p-complete (reduction from SAT for quantified boolean formulas). $\square$

Premise-based Procedure

Definition (PBP)

If $\Phi = \Phi_p \uplus \Phi_c$ is divided into premises and conclusions. The premise-based procedure aggregates a profile $\mathbf{J}$ to a judgment set $\Delta \cup \Gamma$ where:

- $\Phi_p \supseteq \Delta = \{\varphi \in \Phi_p \mid \#\{i \mid \varphi \in J_i\} > \frac{n}{2}\}$
- $\Phi_c \supseteq \Gamma = \{\varphi \in \Phi_c \mid \Delta \models \varphi\}$

We assume Φ_p to be the set of literals occurring in formulas of Φ.

Theorem (easy proof)

$\textsc{WinDet}$(PBP) *is in P.*

Kornhauser and Sager. The one and the many... California Law Review, 1993.
Dietrich and Mongin. The premiss-based approach to JA. JET, 2010.
Endriss, Grandi and Porello. Complexity of WD and strategic manipulation in JA. COMSOC 2010.

Distance-based Procedure

Hamming Distance

If J, J' are two complete and complement-free judgment sets, the Hamming distance $H(J, J')$ is the number of positive formulas on which they differ.

Definition (DBP)

Given an agenda Φ, the distance-based procedure DBP is the function mapping each profile $\mathbf{J} = (J_1, \ldots, J_n)$ to the following set of judgment sets:

$$\mathrm{DBP}(\mathbf{J}) = \text{arg} \min_{J \in J(\Phi)} \sum_{i=1}^{n} H(J, J_i)$$

Theorem

$\textsc{WinDet}^\star$(DBP) *is NP-complete (by reduction to Kemeny-score).*

Konieczny and Pérez. Merging information under constraints: A logical framework. JLC, 2002.

Pigozzi. Belief merging and the discursive dilemma. Synthese, 2006.

Strategic Manipulation

Manipulation in voting theory: *A player can manipulate a voting rule when there exists a situation in which misrepresent her preferences result in an outcome that she prefers to the current one.*

We need a notion of individual preference in JA:

$$J \geqslant_i J' \text{ iff } H(J_i, J) \leqslant H(J_i, J')$$

Manipulability

A JA procedure F is said to be manipulable by agent i at profile $\mathbf{J} = (J_1, \dots, J_i, \dots, J_n)$ if there exist an alternative judgment set $J_i' \in J(\Phi)$ such that $H(J_i, F(J_i', \mathbf{J}_{-i})) < H(J_i, F(\mathbf{J}))$.

Dietrich and List, Strategy-proof judgment aggregation. Economics and Philosophy, 2007.

We can now define the following decision problem:

$\textsc{Manipulable}(F)$	
Instance:	Agenda Φ, judgment set J_i, partial profile $\mathbf{J}_{-i}$.
Question:	Is there a J_i' s.t. $H(J_i, F(J_i', \mathbf{J}_{-i})) < H(J_i, F(J_i, \mathbf{J}_{-i}))$?

Theorem (reduction from $\textsc{sat}$)

$\textsc{Manipulability}(\mathrm{PBP})$ is NP-complete.

Conjecture (hardness)

$\textsc{Manipulability}(\mathsf{DBP}^t)$ is Σ_2^p-complete.

Preference Aggregation and Judgment Aggregation

Arrow's Theorem must have something to do with all these impossibilities...

Definition

Given a finite set of alternatives $\mathcal{X}$, call a preference agenda the following set of atomic formulas $\{aPb \mid a, b \in \mathcal{X}\}$

An individual accepts aPb if she prefers alternative a to b. To enforce individual rationality (i.e. weak orders) we have to add some formulas and assume they are accepted by every individual:

First-order Logic	Propositional Logic
$\forall x.\forall y.xPy \rightarrow \neg yPx$	$aPb \rightarrow \neg aPb \mid a \neq b \in \mathcal{X}$
$\forall x.\forall y.\forall z.xPy \wedge yPz \rightarrow xPz$	$aPc \wedge bPc \rightarrow aPc \mid a, b, c \in \mathcal{X}$
$\forall x.\forall y.xPy \vee yPx$	$aPb \vee bPa \mid a, b \in \mathcal{X}$

Dietrich and List, Arrow's Theorem in Judgment Aggregation. SCW, 2007.

Arrow's Theorem and JA

The two frameworks are equivalent. Arrow's Theorem implies its JA analogous:

Proposition

There exist no judgment aggregation procedure defined on a preference agenda satisfying (A) and (I) (in a slightly modified form).

On the other hand, the "decisiveness" and the "contraction" lemma in the proof of Arrow's Theorem can be generalised to agendas of a specific form:

Proposition

If the agenda Φ is totally blocked and has at least one pair-negatable minimal inconsistent subset, then every aggregation procedure for Φ that satisfies (WR), (U) *and* (I) *is a dictatorship.*

Arrow's Theorem comes as corollary: preference agendas have these properties.

List and Polak, Introduction to Judgment Aggregation. JET, 2010.

Porello, Ranking Judgments in Arrow's Setting. Synthese, 2009.

Yet there is more on this...

Weak orders can be seen as judgment sets over implicative agendas of multi-valued logic, using their representation as utility functions. This embed preference aggregation into judgment aggregation for multi-valued logic.

$$\mathsf{PA}^{wo} \quad \longleftrightarrow \quad \mathsf{JA}^{\rightarrow}_{[0,1]}$$

A judgment set is a dichotomous preference over formulas in the agenda: those being accepted are preferred over those being rejected. This embed judgment aggregation into preference aggregation for (a subclass of) dichotomous preferences.

$$\mathsf{PA}^{\mathsf{dic}} \quad \longleftrightarrow \quad \mathsf{JA}$$

Impossibility theorems have their correspondent on both sides of the arrows.

The rest is an ongoing discussion (in Italian)...

Grossi, Correspondences in the Theory of Aggregation. LOFT 2010.

Last slide

- Everything starts with a paradox in legal doctrine: majority vote on interrelated propositions is inconsistent.
- This has been generalised to several impossibility theorems for judgment aggregation and agenda characterisation results.
- The COMSOC perspective (in Amsterdam):
  - Study the complexity of checking agenda properties, of winner determination and of manipulation of certain aggregation rules.
  - Understand the obscure relation between judgment/preference and binary aggregation.