Computational Social Choice: Autumn 2011

Ulle Endriss
Institute for Logic, Language and Computation
University of Amsterdam

Plan for Today

To be able to analyse, for instance, the incentives of voters to manipulate an election when an irresolute voting rule is used, we need to understand their preferences over *sets of alternatives*.

Today, we will isolate this problem and study it in its own right:

- Problem formulation: *ranking sets of objects*
- The classical result in the field: the *Kannai-Peleg Theorem* (1984)
- Brief mentioning of applications of *automated reasoning* tools

Ranking Sets of Objects

Consider the following problem:

- Given: preference order $\succeq$ declared over *elements* of set $\mathcal{X}$.
- Question: what can we say about the corresponding preference order $\hat{\succeq}$ over *nonempty subsets* of $\mathcal{X}$?
  - Answer: in general, nothing
  - But what are *reasonable principles* for extending preferences?

Besides manipulation in voting theory, other applications include decision making under complete uncertainty.

S. Barberà, W. Bossert, and P.K. Pattanaik. Ranking sets of objects. In *Handbook of Utility Theory*, volume 2. Kluwer Academic Publishers, 2004.

Formal Framework

Three components:

- finite set $\mathcal{X}$ of alternatives (objects, options, candidates, ...)
- total order $\succeq$ on $\mathcal{X}$ (preferences over alternatives)
- weak order $\hat{\succeq}$ on $2^{\mathcal{X}} \setminus \{\emptyset\}$ (preferences over nonempty sets)

preorder:	reflexive, transitive
weak order:	reflexive, transitive, complete
total order:	reflexive, transitive, complete, antisymmetric
linear order:	strict part of total order

Some questions you may ask:

- If we only know $\mathcal{X}$ and $\succeq$, what properties of $\hat{\succeq}$ should we *reasonably* be able to infer?
- What are interesting axioms to impose on structures $\langle \mathcal{X}, \succeq, \hat{\succeq} \rangle$?

Examples

- You know $a \succ b \succ c$.
  Can you infer $\{a\} \mathrel{\hat{\succ}} \{b, c\}$?
- You know $a \succ b \succ c$.
  Can you infer anything regarding $\{b\}$ and $\{a, c\}$?
- You know $a \succ b \succ c \succ d$.
  Can you infer $\{a, b, d\} \mathrel{\hat{\succeq}} \{a, c, d\}$?

Interpretations

Note that there are different possible interpretations to sets of objects:

- You will get one of the elements, but cannot control which.
- You can choose one of the elements.
- You will get the full set.

The first interpretation is usually appropriate for voting, though an optimistic voter might adopt the second interpretation.

Revisit the examples in view of these possible interpretations ...

Notation

For any preference order $\succeq$ (e.g., a preorder), we define:

- its *strict* part: $x \succ y$ if $x \succeq y$ but not $y \succeq x$
- its *indifference* part: $x \sim y$ if both $x \succeq y$ and $y \succeq x$

Kelly Principle

The extension axiom:

(EXT) $\{a\} \mathrel{\hat{\succ}} \{b\}$ if $a \succ b$

Two further axioms:

(MAX) $\{\max(A)\} \mathrel{\hat{\succeq}} A$ $[\max(A) =$ best element in A wrt. $\succ]$

(MIN) $A \mathrel{\hat{\succeq}} \{\min(A)\}$ $[\min(A) =$ worst element in A wrt. $\succ]$

The *Kelly Principle* = (EXT) + (MAX) + (MIN). That is:

- $A \mathrel{\hat{\succ}} B$ if all elements in A are strictly better than all those in B
- $A \mathrel{\hat{\succeq}} B$ if all elements in A are at least as good as all those in B

J.S. Kelly. Strategy-Proofness and Social Choice Functions without Single-Valuedness. *Econometrica*, 45(2):439–446, 1977.

Interpretation in Terms of Tie-Breaking Rules

A possible justification for the Kelly Principle:

- A voter is considering to manipulate an irresolute voting rule. Voting one way will produce winning set A, voting another way will produce B. Tie-breaking will be used to pick one winner.
- Suppose we know the voter's preference order $\succ$ over individual alternatives, but we do not know anything about, say, her preference intensities. We also don't know anything about her beliefs regarding the tie-breaking rule (e.g., probabilities).
- When can we be *sure* that our voter will prefer set A over set B?

  Answer: Exactly when $A \mathrel{\hat{\succ}} B$ according to the Kelly Principle!

Gärdenfors Principle

Two axioms:

(GF1) $A \cup \{b\} \mathrel{\hat{\succ}} A$ if $b \succ a$ for all $a \in A$

(GF2) $A \mathrel{\hat{\succ}} A \cup \{b\}$ if $a \succ b$ for all $a \in A$

The *Gärdenfors Principle* = (GF1) + (GF2):

If I can get from A to B by means of a (nonempty) sequence of steps, each involving deleting the best element or adding a new worst element, then A is strictly better than B.

The Gärdenfors Principle *entails* the Kelly Principle, but not *vice versa*.

P. Gärdenfors. Manipulation of Social Choice Functions. *Journal of Economic Theory*. 1392):217–228, 1976.

Rational Tie-Breaking

We can also interpret the Gärdenfors Principle by means of tie-breaking rules in voting:

- Suppose we know that ties will be broken by somebody who is *rational* in the sense that she is endowed with a linear order over $\mathcal{X}$ and will always pick the element that is maximal wrt. that order.

  That is: we know that some choice functions (e.g., that depend on context or that are probabilistic) are excluded.
- Then we can be *sure* that our voter would prefer A over B exactly when $A \mathrel{\hat{\succ}} B$ according to the Gärdenfors Principle.

Independence

The *independence* axiom:

(IND) $A \cup \{c\} \mathrel{\hat{\succeq}} B \cup \{c\}$ if $A \mathrel{\hat{\succ}} B$ and $c \notin A \cup B$

That is: if you (strictly) prefer A over B, then that preference should not get inverted when we add a new object c to both sets.

The Kannai-Peleg Theorem

The 1984 paper by Yakar Kannai and Bezalel Peleg is considered *the* seminal contribution to the axiomatic study of *ranking sets of objects*.

Theorem 1 (Kannai and Peleg, 1984) *If $|\mathcal{X}| \geqslant 6$, then no weak order $\hat{\succeq}$ satisfies both the Gärdenfors Principle and independence.*

Probably the first paper treating the problem of preference extension as a *problem in its own right*, from an *axiomatic* perspective.

- For Kelly and Gärdenfors (and others), the problem has been more of a side issue (when studying manipulation in voting).
- Work on the problem of ranking sets of objects itself published before 1984 is descriptive rather than axiomatic.

Y. Kannai and B. Peleg. A Note on the Extension of an Order on a Set to the Power Set. *Journal of Economic Theory*, 32(1):172–175, 1984.

Lemma

Recall the axioms:

$$\left.\begin{array}{ll}\text{(GF1)}\ A \cup \{b\} \hat{\succ} A & \text{if } b \succ a \text{ for all } a \in A \\ \text{(GF2)}\ A \hat{\succ} A \cup \{b\} & \text{if } a \succ b \text{ for all } a \in A\end{array}\right\} \text{Gärdenfors Principle}$$

$$\text{(IND)}\ A \cup \{c\} \hat{\succeq} B \cup \{c\} \quad \text{if } A \hat{\succ} B \text{ and } c \notin A \cup B$$

Lemma 1 *Gärdenfors + (IND) entails $A \hat{\sim} \{\max(A), \min(A)\}$.*

Proof:

- If $|A| \leqslant 2$, then $A = \{\max(A), \min(A)\}$. ✓
- If $|A| > 2$:
  - $A \setminus \{\max(A)\} \hat{\succ} \{\min(A)\}$ by repeated application of (GF1), and thus $A \hat{\succeq} \{\max A, \min(A)\}$ by (IND).
  - $\{\max(A)\} \hat{\succ} A \setminus \{\min(A)\}$ by repeated application of (GF2), and thus $\{\max(A), \min(A)\} \hat{\succeq} A$ by (IND).

  Hence, $A \hat{\sim} \{\max(A), \min(A)\}$. ✓

Proof of the Kannai-Peleg Theorem

Theorem: If $|\mathcal{X}| \geqslant 6$, then no weak order $\hat{\succeq}$ satisfies both the Gärdenfors Principle and independence.

Proof: Suppose $a_6 \succ a_5 \succ a_4 \succ a_3 \succ a_2 \succ a_1$.

Claim: $\{a_2, a_5\} \hat{\succeq} \{a_4\}$ $(*)$

Proof of claim: if not, then $\{a_4\} \hat{\succ} \{a_2, a_5\}$, as $\hat{\succeq}$ is complete

$\Rightarrow \{a_1, a_4\} \hat{\succeq} \{a_1, a_2, a_5\}$ by (IND)

$\Rightarrow \{a_1, a_2, a_3, a_4\} \hat{\succeq} \{a_1, a_2, a_3, a_4, a_5\}$ by Lemma $\Rightarrow \lightning$ [GP]

Hence: $\{a_2, a_5\} \hat{\succ} \{a_3\}$ from $(*)$ and $\{a_4\} \hat{\succ} \{a_3\}$ [GP]

$\Rightarrow \{a_2, a_5, a_6\} \hat{\succeq} \{a_3, a_6\}$ by (IND)

$\Rightarrow \{a_2, a_3, a_4, a_5, a_6\} \hat{\succeq} \{a_3, a_4, a_5, a_6\}$ by Lemma $\Rightarrow \lightning$ [GP]

Done. ✓

Remark: Note that there *are preorders* satisfying all axioms, e.g.:

$A \hat{\succeq} B :\Leftrightarrow \max(A) \succeq \max(B)$ and $\min(A) \succeq \min(B)$

Automated Theorem Search

One of the major challenges in computational social choice is to formally model social choice problems so as to facilitate the automated verification, and possibly even the automated discovery, of theorems.

For the relatively simple domain of *ranking sets of objects*, this is indeed possible (MoL thesis by Christian Geist, 2010):

- Introduce a logic for expressing axioms (many-sorted FOL).
- Identify syntactic conditions on axioms under which any impossibility verified for $|\mathcal{X}| = k$ generalises to all larger domains.
- For a fixed domain, axioms can be expressed in propositional logic.
- Impossibility for a fixed domain can be checked by a SAT solver.
- Can search over all combinations of axioms from a given set and thereby discover all impossibilities (found 84 impossibility theorems for 20 axioms, that apply to any domain $\mathcal{X}$ with $|\mathcal{X}| \geqslant 8$).

C. Geist and U. Endriss. Automated Search for Impossibility Theorems in Social Choice Theory: Ranking Sets of Objects. *J. Artif. Intell. Res.*, 40:143–174, 2011.

Summary

This has been an introduction to the field of *ranking sets of objects*.

- Different interpretations to the formal problem $\Rightarrow$ different axioms
- Basic principles of preference extension: Kelly, Gärdenfors
- Problematic axiom: independence
- Impossibility theorem: Kannai-Peleg

We have also seen an example for work on using logic and automated reasoning to derive new results in this domain.

What next?

We will take up several of the issues touched on today in later lectures:

- Today we have hinted at the fact that formalising an area of social choice theory in logic can have useful applications. We will discuss the (limited) progress that has so far been made along these lines in a lecture on *logics for social choice*.
- Today we tried to *infer* a preference order on sets purely from the given order on objects. Later we will also discuss how to *define* orders on sets (lecture on *compact preference representation*).