Plan for Today

Today's lecture will be devoted to classical impossibility theorems in social choice theory. We already proved Arrow's Theorem using the "decisive coalition" technique. Today we'll first review this result and:

- give references to alternative proofs
- discuss the challenge of automatically proving Arrow's Theorem

Then we'll see two further classical impossibility theorems:

- Sen's Theorem on the Impossibility of a Paretian Liberal (1970)
- the Muller-Satterthwaite Theorem (1977)

The former is easy to prove; for the latter we will again use the "decisive coalition" technique.

Arrow's Theorem

Recall terminology and axioms:
- SWF: $F : L(X)^N \rightarrow L(X)$
- Pareto: $N_{x \succ y} \Rightarrow (x, y) \in F(R)$
- IIA: $N_{x \succ y} = N_{x' \succ y} \Rightarrow (x, y) \in F(R) \Leftrightarrow (x, y) \in F(R')$
- Dictatorship: $\exists i \in N$ s.t. $\forall (R_1, \ldots, R_n): F(R_1, \ldots, R_n) = R_i$

Here is again the theorem:

**Theorem 1 (Arrow, 1951)** Any SWF for $\geq 3$ alternatives that satisfies the Pareto condition and IIA must be a dictatorship.


Alternative Proofs

Arrow's book is an inspiring and interesting read, but his proof is very verbose and hard to follow (and the original version of 1951 famously has a small mistake in the theorem). Some alternative proofs:

- Geanakoplos (2005) gives three short proofs. The first one is particularly helpful. It uses the "pivotal voter" technique and is based on earlier work by Barberà (1980).

- Another proof involves showing that the family of decisive coalitions is an ultrafilter for $N$ (Kirman and Sondermann, 1972).


Automated Reasoning for Social Choice Theory

There've also been attempts to automate proving Arrow’s Theorem:

- Nipkow (2009) has encoded one of Geanakoplos’ proofs in the language of the higher-order logic proof assistant Isabelle, resulting in an automatic verification of that proof.
- Tang and Lin (2009) have translated Arrow’s Theorem for 2 individuals and 3 alternatives into a set of clauses in propositional logic, which allows for verification by means of a SAT-solver.


Logics for Social Choice Theory

More generally, it is interesting to explore the use of logics to model problems studied in SCT. This is still an under-developed strand of research. Below are some references (the last one is a survey).


Social Choice Functions

From now on we consider aggregators that take a profile of preferences and return one or several “winners” (rather than a full social ranking). This is called a social choice function (SCF):

\[ F : \mathcal{L}(X)^N \rightarrow 2^X \setminus \{\emptyset\} \]

A SCF is called resolute if \( |F(R)| = 1 \) for any given profile \( R \), i.e., if it always selects a unique winner.

Remark: We can think of a SCF as a voting rule, particularly if it tends to select “small” sets of winners (we won’t make this precise). Voting rules are often required to be resolute (¬ tie-breaking rule).

Examples: The plurality and the Borda rule are both (irresolute) SCFs. Approval voting is not a SCF (inputs are sets, not linear orders).

Alternative Definition

In the literature you will sometimes find the term SCF being used for functions \( F : \mathcal{L}(X)^N \times 2^X \setminus \{\emptyset\} \rightarrow 2^X \setminus \{\emptyset\} \). Two readings:

- The input of \( F \) is a profile of preferences (as before) + a set of feasible alternatives. The output should be a subset of the feasible alternatives, selected in view of the preference profile.
- The input of \( F \) is just a profile of preferences (as before). The output is a choice function \( C : 2^X \setminus \{\emptyset\} \rightarrow 2^X \setminus \{\emptyset\} \) that will select a set of winners from any given set of alternatives.

This refinement is not relevant for the results we want to discuss here, so we shall take a SCF to be a function \( F : \mathcal{L}(X)^N \rightarrow 2^X \setminus \{\emptyset\} \).
The Pareto Condition for Social Choice Functions

A SCF $F$ satisfies the Pareto condition if, whenever all individuals rank $x$ above $y$, then $y$ cannot win:

$$N^R_{x,y} = N \implies y \notin F(R)$$

Liberalism

Think of $X$ as the set of all possible “social states”. Certain aspects of such a state will be some individual’s private business. Example:

If $x$ and $y$ are identical states, except that in $x$ I paint my bedroom white, while in $y$ I paint it pink, then I should be able to dictate the relative social ranking of $x$ and $y$.

Sen (1970) proposed the following axiom:

A SCF $F$ satisfies the axiom of liberalism if, for every individual $i \in N$, there exist two distinct alternatives $x, y \in X$ such that $i$ is two-way decisive on $x$ and $y$:

$$i \in N^R_{x,y} \implies y \notin F(R) \text{ and } i \in N^R_{y,x} \implies x \notin F(R)$$

The Impossibility of a Paretian Liberal

Sen (1970) showed that liberalism and the Pareto condition are incompatible (recall that we required $|N| \geq 2$, which matters here):

**Theorem 2 (Sen, 1970)** No SCF satisfies both liberalism and the Pareto condition.

As we shall see, the theorem holds even when liberalism is enforced for only two individuals. The number of alternatives does not matter.

Again, a surprising result (but easier to prove than Arrow’s Theorem).

Proof

Let $F$ be a SCF satisfying Pareto and liberalism. Get a contradiction:

Take two distinguished individuals $i_1$ and $i_2$, with:

- $i_1$ is two-way decisive on $x_1$ and $y_1$
- $i_2$ is two-way decisive on $x_2$ and $y_2$

Assume $x_1, y_1, x_2, y_2$ are pairwise distinct (other cases: easy).

Consider a profile with these properties:

1. Individual $i_1$ ranks $x_1 \succ y_1$.
2. Individual $i_2$ ranks $x_2 \succ y_2$.
3. All individuals rank $y_1 \succ x_2$ and $y_2 \succ x_1$.
4. All individuals rank $x_1, x_2, y_1, y_2$ above all other alternatives.

From liberalism: (1) rules out $y_1$ and (2) rules out $y_2$ as winner.

From Pareto: (3) rules out $x_1$ and $x_2$ and (4) rules out all others.

Thus, there are no winners. Contradiction. ✓
Monotonicity

Next we want to formalise the idea that when a winner receives increased support, she should not become a loser.

We focus on resolute SCFs. Write $x^* = F(R)$ for $\{x^*\} = F(R)$.

- **Weak monotonicity**: $F$ is weakly monotonic if $x^* = F(R)$ implies $x^* = F(R')$ for any alternative $x^*$ and any two profiles $R$ and $R'$ with $N_{x^*>y}^R \subseteq N_{x^*>y}^{R'}$ and $N_{y^*>z}^R = N_{y^*>z}^{R'}$ for all $y, z \in X \setminus \{x^*\}$.

- **Strong monotonicity**: $F$ is strongly monotonic if $x^* = F(R)$ implies $x^* = F(R')$ for any alternative $x^*$ and any two profiles $R$ and $R'$ with $N_{x^*>y}^R \subseteq N_{x^*>y}^{R'}$ for all $y \in X \setminus \{x^*\}$.

The latter property is also known as Maskin monotonicity or strong positive association.

Example

Even weak monotonicity is not satisfied by some common voting rules. Under plurality with runoff the two alternatives with the highest plurality score enter a second round and the majority winner of that round is the winner (used to elect the French president). Example:

| 27 voters: | 42 voters: | 24 voters: |
| A > B > C | C > A > B | B > C > A |

$B$ is eliminated in the first round and $C$ beats $A$ 66:27 in the runoff. But if 4 of the voters in the first group raise $C$ to the top (i.e., join the second group), then $B$ will win.

But many other rules (e.g., plurality) do satisfy weak monotonicity. How about strong monotonicity?

Proof

We use again the "decisive coalition" technique. Full details are available in the review paper cited below.

Claim: Any resolute SCF for $\geq 3$ alternatives that is surjective and strongly monotonic must be a dictatorship.

Let $F$ be a SCF for $\geq 3$ alt. that is surjective and strongly monotonic.

Proof Plan:

- Show that $F$ must be independent (to be defined).
- Show that $F$ must be Pareto efficient.
- Prove a version of Arrow's Theorem for SCFs.
Independence

Call a SCF $F$ independent if it is the case that $x \neq y$, $F(R) = x$, and $N_{x>y}^R = N_{y>x}^R$ together imply $F(R') \neq y$.

That is, if $y$ lost to $x$ under profile $R$, and the relative rankings of $x$ vs. $y$ do not change, then $y$ will still lose (possibly to a different winner).

Claim: $F$ is independent.

Proof: Suppose $x \neq y$, $F(R) = x$, and $N_{x>y}^R = N_{y>x}^R$.

Construct a third profile $R'$:

- All individuals rank $x$ and $y$ in the top-two positions.
- The relative rankings of $x$ vs. $y$ are as in $R$, i.e., $N_{x>y}^{R'} = N_{y>x}^R$.
- Rest: whatever

By strong monotonicity, $F(R) = x$ implies $F(R') = x$.

By strong monotonicity, $F(R') = y$ would imply $F(R') = y$.

Thus, we must have $F(R') \neq y$. ✓

Pareto Condition

Recall the Pareto condition: if everyone ranks $x \succ y$, then $y$ won’t win.

Claim: $F$ satisfies the Pareto condition.

Proof: Take any two alternatives $x$ and $y$.

From surjectivity: $x$ will win for some profile $R$.

Starting in $R$, have everyone move $x$ above $y$ (if not above already).

From strong monotonicity: $x$ still wins.

From independence: $y$ does not win for any profile where all individuals continue to rank $x \succ y$. ✓

Plan for the Rest of the Proof

We now know that $F$ must be a SCF for $\geq 3$ alternatives that is independent and Pareto efficient. We want to infer that $F$ must be a dictatorship.

Call a coalition $G \subseteq \mathcal{N}$ decisive on $(x, y)$ iff $G \subseteq N_{x>y}^R \Rightarrow y \neq F(R)$.

Proof plan:

- Pareto condition = $\mathcal{N}$ is decisive for all pairs of alternatives
- Lemma: $G$ with $|G| \geq 2$ decisive for all pairs $\Rightarrow$ some $G' \subset G$ as well
- Thus (by induction), there’s a decisive coalition of size 1 (a dictator).

About Decisiveness

Recall: $G \subseteq \mathcal{N}$ decisive on $(x, y)$ iff $G \subseteq N_{x>y}^R \Rightarrow y \neq F(R)$

Call $G \subseteq \mathcal{N}$ weakly decisive on $(x, y)$ iff $G = N_{x>y}^R \Rightarrow y \neq F(R)$.

Claim: $G$ weakly decisive on $(x, y) \Rightarrow G$ decisive on any pair $(x', y')$

Proof: Suppose $x, y, x', y'$ are all distinct (other cases: similar).

Consider a profile where individuals express these preferences:

- Members of $G$: $x' \succ x \succ y \succ y'$
- Others: $x' \succ x$, $y \succ y'$, and $y \succ x$ (note that $x'$-vs.-$y'$ is not specified)
- All rank $x, y, x', y'$ above all other alternatives.

From $G$ being weakly decisive for $(x, y)$, $y$ must lose

From Pareto $\Rightarrow x$ must lose (to $x'$) and $y'$ must lose (to $y$)

Thus, $x'$ must win (and $y'$ must lose). By independence, $y'$ will still lose when everyone changes their non-$x'$-vs.-$y'$ rankings.

Thus, for any profile $R$ with $G \subseteq N_{x'y'}^R$, we get $y' \neq F(R)$. ✓
**Contraction Lemma**

**Claim:** If \( G \subseteq N \) with \( |G| \geq 2 \) is a coalition that is decisive on all pairs of alternatives, then so is some nonempty coalition \( G' \subset G \).

**Proof:** Take any nonempty \( G_1, G_2 \) with \( G = G_1 \cup G_2 \) and \( G_1 \cap G_2 = \emptyset \).

Recall that there are \( \geq 3 \) alternatives. Consider this profile:

- Members of \( G_1 \): \( x \succ y \succ z \succ \text{rest} \)
- Members of \( G_2 \): \( y \succ z \succ x \succ \text{rest} \)
- Others: \( z \succ x \succ y \succ \text{rest} \)

As \( G = G_1 \cup G_2 \) is decisive, \( z \) cannot win (it loses to \( y \)). Two cases:

1. The winner is \( x \): Exactly \( G_1 \) ranks \( x \succ z \Rightarrow \) By independence, in any profile where exactly \( G_1 \) ranks \( x \succ z \), \( z \) will lose (to \( x \)) \( \Rightarrow G_1 \) is weakly decisive on \( (x, z) \). Hence (previous slide): \( G_1 \) is decisive on all pairs.

2. The winner is \( y \), i.e., \( x \) loses (to \( y \)). Exactly \( G_2 \) ranks \( y \succ x \Rightarrow \cdots \Rightarrow G_2 \) is decisive on all pairs.

Hence, one of \( G_1 \) and \( G_2 \) will always be decisive. \( \checkmark \)

**Summary**

We have by now see three important impossibility theorems, establishing the incompatibility of certain desirable properties:

- **Arrow:** Pareto, IIA, nondictatoriality
- **Sen:** Pareto, liberalism
- **Muller-Satterthwaite:** surjectivity, strong monotonicity, nondictat.

We have discussed these results in two formal frameworks (none of the results heavily depend on the choice of framework):

- social welfare functions (SWF)
- (resolute) social choice functions (SCF)

This has also been an introduction to the [axiomatic method]:

- formulate desirable properties of aggregators as axioms
- explore the consequences of imposing several such axioms

**What next?**

As discussed, the impossibility theorems we have seen can also be interpreted as axiomatic characterisations of the class of dictatorships.

Soon we will see characterisations of more attractive (classes of) voting rules:

- using (again) the axiomatic method; and
- using different methods.

But first we will see more examples for practical voting rules and discuss their properties, including also their algorithmic properties (how hard is it to compute the winners?).