Computational Social Choice: Autumn 2012

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Plan for Today

We will introduce a few (more) voting rules:

- Staged procedures
- Positional scoring rules
- Condorcet extensions

And we will discuss some of their properties, including these:

- the Condorcet principle
- the computational complexity of the problem of determining the winner of an election

This discussion will give some initial guidelines for choosing a suitable voting rule for a specific situation at hand (an intricate problem that we won’t fully resolve).
Many Voting Rules

There are many different voting rules. Many, not all, of them are defined in the survey paper by Brams and Fishburn (2002).

Most voting rules are social choice functions:

- Borda, Plurality, Antiplurality/Veto, and $k$-approval, Plurality with Runoff, Single Transferable Vote (STV), Nanson, Cup Rule/Voting Trees, Copeland, Banks, Slater, Schwartz, Minimax/Simpson, Kemeny, Ranked Pairs/Tideman, Schulze, Dodgson, Young, Bucklin.

But some are not:


Single Transferable Vote (STV)

STV (also known as the *Hare system*) is a *staged procedure*:

- If one of the candidates is the 1st choice for over 50% of the voters (*quota*), she wins.
- Otherwise, the candidate who is ranked 1st by the fewest voters gets *eliminated* from the race.
- Votes for eliminated candidates get *transferred*: delete removed candidates from ballots and “shift” rankings (i.e., if your 1st choice got eliminated, then your 2nd choice becomes 1st).

In practice, voters need not be required to rank all candidates (non-ranked candidates are assumed to be ranked lowest).

STV is used in several countries (e.g., Australia, New Zealand, . . . ).

For three candidates, STV and Plurality with Runoff coincide.

**Variants:** Coombs, Nanson, Baldwin
The No-Show Paradox

Under plurality with runoff (and thus under STV), it may be better to abstain than to vote for your favourite candidate! Example:

- 25 voters: $A \succ B \succ C$
- 46 voters: $C \succ A \succ B$
- 24 voters: $B \succ C \succ A$

Given these voter preferences, $B$ gets eliminated in the first round, and $C$ beats $A$ 70:25 in the runoff.

Now suppose two voters from the first group abstain:

- 23 voters: $A \succ B \succ C$
- 46 voters: $C \succ A \succ B$
- 24 voters: $B \succ C \succ A$

$A$ gets eliminated, and $B$ beats $C$ 47:46 in the runoff.

**Positional Scoring Rules**

We can generalise the idea underlying the Borda rule as follows:

A *positional scoring rule* is given by a *scoring vector* $s = \langle s_1, \ldots, s_m \rangle$ with $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$.

Each voter submits a ranking of the $m$ alternatives. Each alternative receives $s_i$ points for every voter putting it at the $i$th position.

The alternatives with the highest score (sum of points) win.

**Examples:**

- **Borda rule** = PSR with scoring vector $\langle m-1, m-2, \ldots, 0 \rangle$
- **Plurality rule** = PSR with scoring vector $\langle 1, 0, \ldots, 0 \rangle$
- **Antiplurality rule** = PSR with scoring vector $\langle 1, \ldots, 1, 0 \rangle$
- For any $k \leq m$, **$k$-approval** = PSR with $\langle 1, \ldots, 1, 0, \ldots, 0 \rangle^k$
The Condorcet Principle

An alternative that beats every other alternative in pairwise majority contests is called a Condorcet winner.

There may be no Condorcet winner; witness the Condorcet paradox:

<table>
<thead>
<tr>
<th>Ann</th>
<th>Bob</th>
<th>Cesar</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \succ B \succ C$</td>
<td>$B \succ C \succ A$</td>
<td>$C \succ A \succ B$</td>
</tr>
</tbody>
</table>

Whenever a Condorcet winner exists, then it must be unique.

A voting procedure satisfies the Condorcet principle if it elects (only) the Condorcet winner whenever one exists.

Positional Scoring Rules violate Condorcet

Consider the following example:

3 voters: \( A \succ B \succ C \)
2 voters: \( B \succ C \succ A \)
1 voter: \( B \succ A \succ C \)
1 voter: \( C \succ A \succ B \)

\( A \) is the \textit{Condorcet winner}; she beats both \( B \) and \( C \) 4:3. But any \textit{positional scoring rule} makes \( B \) win (because \( s_1 \geq s_2 \geq s_3 \)):

\[
\begin{align*}
A & : \ 3 \cdot s_1 + 2 \cdot s_2 + 2 \cdot s_3 \\
B & : \ 3 \cdot s_1 + 3 \cdot s_2 + 1 \cdot s_3 \\
C & : \ 1 \cdot s_1 + 2 \cdot s_2 + 4 \cdot s_3
\end{align*}
\]

Thus, \textit{no positional scoring rule} for three (or more) alternatives will satisfy the \textit{Condorcet principle}.
Condorcet Extensions

A *Condorcet extension* is a voting rule that respects the Condorcet principle. Fishburn suggested the following classification:

- **C1**: Rules for which the winners can be computed from the *majority graph* alone. Example:
  - *Copeland*: elect the candidate that maximises the difference between won and lost pairwise majority contests
- **C2**: Non-C1 rules for which the winners can be computed from the *weighted majority graph* alone. Example:
  - *Kemeny*: elect top candidates in rankings that minimise sum of Hamming distances to individual rankings
- **C3**: All other Condorcet extensions. Example:
  - *Young*: elect candidates that minimise number of voters to be removed before they become Condorcet winners

Complexity of Winner Determination

Bartholdi et al. (1989) were the first to study the complexity of computing election winners. They showed that checking whether a candidate’s *Dodgson* score exceeds $K$ is NP-complete. Other results include:

- Checking whether a candidate is a Dodgson winner it is *complete for parallel access to NP* (Hemaspaandra et al., 1997). There are similar results for the Kemeny rule. Young and Slater are also hard.

- More recent work has also analysed the *parametrised complexity* of winner determination. See Betzler et al. (2012) for a good introduction.


The Banks Rule

Let $\mathcal{X}$ be the set of alternatives. Define the majority graph $(\mathcal{X}, \succ_M)$:

$$x \succ_M y \text{ iff a strict majority of voters rank } x \text{ above } y$$

Aside: If $(\mathcal{X}, \succ_M)$ is complete, then it is called a tournament. That is, if the number $n$ of voters is odd, then $(\mathcal{X}, \succ_M)$ is a tournament.

Under the Banks rule, a candidate $x$ is a winner if it is a top element in a maximal acyclic subgraph of the majority graph.

Fact: The Banks rule respects the Condorcet principle.

Complexity of Winner Determination: Banks Rule

A desirable property of any voting rule is that it should be easy (computationally tractable) to compute the winner(s).

For the Banks rule, we formulate the problem wrt. the majority graph (which we can compute in polynomial time given the ballot profile):

**Banks-Winner**

**Instance:** majority graph $G = (X, \succ_M)$ and alternative $x^* \in X$

**Question:** Is $x^*$ a Banks winner for $G$?

Unfortunately, recognising Banks winners is intractable:

**Theorem 1 (Woeginger, 2003)** *Banks-Winner* is NP-complete.

**Proof:** NP-membership: certificate = maximal acyclic subgraph 
NP-hardness: reduction from *Graph 3-Colouring* (see paper). ✓

Easiness of Computing Some Winner

We have seen that checking whether $x$ is a Banks winner is NP-hard. So computing all Banks winners is also NP-hard.

But computing just some Banks winner is easy! Algorithm:

(1) Let $S := \{x_1\}$ and $i := 1$. [candidates $\mathcal{X} = \{x_1, \ldots, x_m\}$]

(2) While $i < m$, repeat:
   • Let $i := i + 1$.
   • If the majority graph restricted to $S \cup \{x_i\}$ is acyclic, then let $S := S \cup \{x_i\}$.

(3) Return the top element in $S$ (it is a Banks winner).

This algorithm has complexity $O(m^2)$ if given the majority graph, which in turn can be constructed in time $O(n \cdot m^2)$.

Summary

We have by now seen several types of voting rules:

- **staged procedures**: STV, Plurality with Runoff, . . .
- **positional scoring rules**: Borda, Plurality, Antiplurality, . . .
- **Condorcet extensions**: Copeland, Banks, Kemeny, Young, . . .

Helpful references for these and other voting rules are the works of Brams and Fishburn (2002) and Nurmi (1987).

We have also discussed three important properties:

- **Participation**: a voting rule should not give incentives not to vote (i.e., it should not suffer from the *no-show paradox*)
- **Condorcet principle**: elect the Condorcet winner whenever it exists
- **Complexity of winner determination**: computing the winner(s) of an election should be computationally tractable


What next?

In the next lecture we will see three different approaches to providing characterisations of voting rules.

- This will provide some explanation for the enormous diversity of voting rules encountered today.
- It will also connect to the impossibility theorems we have seen before, which may be considered characterisations of dictatorships.