# Computational Social Choice: Autumn 2012

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# Plan for Today

We will introduce a few (more) *voting rules:*

- *Staged procedures*
- *Positional scoring rules*
- *Condorcet extensions*

And we will discuss some of their *properties*, including these:

- the *Condorcet principle*
- the *computational complexity* of the problem of determining the *winner* of an election

This discussion will give some initial guidelines for choosing a suitable voting rule for a specific situation at hand (an intricate problem that we won't fully resolve).

# Many Voting Rules

There are many different voting rules. Many, not all, of them are defined in the survey paper by Brams and Fishburn (2002).

Most voting rules are social choice functions:

> Borda, Plurality, Antiplurality/Veto, and $k$-approval, Plurality with Runoff, Single Transferable Vote (STV), Nanson, Cup Rule/Voting Trees, Copeland, Banks, Slater, Schwartz, Minimax/Simpson, Kemeny, Ranked Pairs/Tideman, Schulze, Dodgson, Young, Bucklin.

But some are not:

> Approval Voting, Majority Judgment, Cumulative Voting, Range Voting.

S.J. Brams and P.C. Fishburn. Voting Procedures. In K.J. Arrow et al. (eds.), *Handbook of Social Choice and Welfare*, Elsevier, 2002.

# Single Transferable Vote (STV)

STV (also known as the *Hare system*) is a *staged procedure:*

- If one of the candidates is the 1st choice for over $50\%$ of the voters (*quota*), she wins.
- Otherwise, the candidate who is ranked 1st by the fewest voters gets *eliminated* from the race.
- Votes for eliminated candidates get *transferred:* delete removed candidates from ballots and "shift" rankings (i.e., if your 1st choice got eliminated, then your 2nd choice becomes 1st).

In practice, voters need not be required to rank all candidates (non-ranked candidates are assumed to be ranked lowest).

STV is used in several countries (e.g., Australia, New Zealand, . . . ).

For three candidates, STV and Plurality with Runoff coincide.

Variants: Coombs, Nanson, Baldwin

# The No-Show Paradox

Under plurality with runoff (and thus under STV), it may be better to abstain than to vote for your favourite candidate! Example:

25 voters: $A \succ B \succ C$
46 voters: $C \succ A \succ B$
24 voters: $B \succ C \succ A$

Given these voter preferences, $B$ gets eliminated in the first round, and $C$ beats $A$ 70:25 in the runoff.

Now suppose two voters from the first group abstain:

23 voters: $A \succ B \succ C$
46 voters: $C \succ A \succ B$
24 voters: $B \succ C \succ A$

$A$ gets eliminated, and $B$ beats $C$ 47:46 in the runoff.

P.C. Fishburn and S.J Brams. Paradoxes of Preferential Voting. *Mathematics Magazine*, 56(4):207-214, 1983.

# Positional Scoring Rules

We can generalise the idea underlying the Borda rule as follows:

A *positional scoring rule* is given by a *scoring vector* $s = \langle s_1, \ldots, s_m \rangle$ with $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_m$ and $s_1 > s_m$.

Each voter submits a ranking of the $m$ alternatives. Each alternative receives $s_i$ points for every voter putting it at the $i$th position.

The alternatives with the highest score (sum of points) win.

Examples:

- *Borda rule* = PSR with scoring vector $\langle m-1, m-2, \ldots, 0 \rangle$
- *Plurality rule* = PSR with scoring vector $\langle 1, 0, \ldots, 0 \rangle$
- *Antiplurality rule* = PSR with scoring vector $\langle 1, \ldots, 1, 0 \rangle$
- For any $k \leqslant m$, *$k$-approval* = PSR with $\langle \underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0 \rangle$

# The Condorcet Principle

An alternative that beats every other alternative in pairwise majority contests is called a *Condorcet winner*.

There may be no Condorcet winner; witness the *Condorcet paradox:*

| | |
|---|---|
| Ann: | $A \succ B \succ C$ |
| Bob: | $B \succ C \succ A$ |
| Cesar: | $C \succ A \succ B$ |

Whenever a Condorcet winner exists, then it must be *unique*.

A voting procedure satisfies the *Condorcet principle* if it elects (only) the Condorcet winner whenever one exists.

M. le Marquis de Condorcet. *Essai sur l'application de l'analyse à la probabilté des décisions rendues a la pluralité des voix*. Paris, 1785.

# Positional Scoring Rules violate Condorcet

Consider the following example:

$$
\begin{array}{ll}
\text{3 voters:} & A \succ B \succ C \\
\text{2 voters:} & B \succ C \succ A \\
\text{1 voter:} & B \succ A \succ C \\
\text{1 voter:} & C \succ A \succ B
\end{array}
$$

$A$ is the *Condorcet winner*; she beats both $B$ and $C$ $4:3$. But any *positional scoring rule* makes $B$ win (because $s_1 \geqslant s_2 \geqslant s_3$):

$$
\begin{array}{ll}
A: & 3 \cdot s_1 + 2 \cdot s_2 + 2 \cdot s_3 \\
B: & 3 \cdot s_1 + 3 \cdot s_2 + 1 \cdot s_3 \\
C: & 1 \cdot s_1 + 2 \cdot s_2 + 4 \cdot s_3
\end{array}
$$

Thus, *no positional scoring rule* for three (or more) alternatives will satisfy the *Condorcet principle*.

# Condorcet Extensions

A *Condorcet extension* is a voting rule that respects the Condorcet principle. Fishburn suggested the following classification:

- *C1:* Rules for which the winners can be computed from the *majority graph* alone. Example:
  - *Copeland:* elect the candidate that maximises the difference between won and lost pairwise majority contests
- *C2:* Non-C1 rules for which the winners can be computed from the *weighted majority graph* alone. Example:
  - *Kemeny:* elect top candidates in rankings that minimise sum of Hamming distances to individual rankings
- *C3:* All other Condorcet extensions. Example:
  - *Young:* elect candidates that minimise number of voters to be removed before they become Condorcet winners

P.C. Fishburn. Condorcet Social Choice Functions. *SIAM Journal on Applied Mathematics*, 33(3):469–489, 1977.

# Complexity of Winner Determination

Bartholdi et al. (1989) were the first to study the complexity of computing election winners. They showed that checking whether a candidate's *Dodgson* score exceeds $K$ is NP-complete. Other results include:

- Checking whether a candidate is a Dodgson winner it is *complete for parallel access to NP* (Hemaspaandra et al., 1997). There are similar results for the Kemeny rule. Young and Slater are also hard.
- More recent work has also analysed the *parametrised complexity* of winner determination. See Betzler et al. (2012) for a good introduction.

J.J. Bartholdi III, C.A. Tovey, and M.A. Trick. Voting schemes for which it can be difficult to tell who won the election. *Soc. Choice Welf.*, 6(2):157–165, 1989.

E. Hemaspaandra, L.A. Hemaspaandra, and J. Rothe. Exact Analysis of Dodgson Elections. *Journal of the ACM*, 44(6):806–825, 1997.

N. Betzler, R. Bredereck, J. Chen, and R. Niedermeier. Studies in Computational Aspects of Voting: A Parameterized Complexity Perspective. In *The Multivariate Algorithmic Revolution and Beyond*, pp. 318–363, Springer, 2012.

## The Banks Rule

Let $\mathcal{X}$ be the set of alternatives. Define the *majority graph* $(\mathcal{X}, \succ_M)$:

$$x \succ_M y \text{ } \textit{iff} \text{ a strict majority of voters rank } x \text{ above } y$$

Aside: If $(\mathcal{X}, \succ_M)$ is complete, then it is called a *tournament*. That is, if the number $n$ of voters is odd, then $(\mathcal{X}, \succ_M)$ is a tournament.

Under the *Banks rule*, a candidate $x$ is a winner if it is a top element in a maximal acyclic subgraph of the majority graph.

Fact: The Banks rule respects the Condorcet principle.

J.S. Banks. Sophisticated Voting Outcomes and Agenda Control. *Social Choice and Welfare*, 1(4)295–306, 1985.

# Complexity of Winner Determination: Banks Rule

A desirable property of any voting rule is that it should be easy (computationally tractable) to compute the winner(s).

For the Banks rule, we formulate the problem wrt. the majority graph (which we can compute in polynomial time given the ballot profile):

BANKS-WINNER
**Instance:** majority graph $G = (\mathcal{X}, \succ_M)$ and alternative $x^\star \in \mathcal{X}$
**Question:** Is $x^\star$ a Banks winner for $G$?

Unfortunately, recognising Banks winners is intractable:

**Theorem 1 (Woeginger, 2003)** BANKS-WINNER *is NP-complete.*

Proof: NP-membership: certificate = maximal acyclic subgraph
NP-hardness: reduction from GRAPH 3-COLOURING (see paper). ✓

G.J. Woeginger. Banks Winners in Tournaments are Difficult to Recognize. *Social Choice and Welfare*, 20(3)523–528, 2003.

# Easiness of Computing Some Winner

We have seen that checking whether $x$ is a Banks winner is NP-hard. So computing *all* Banks winners is also NP-hard.

But computing just *some* Banks winner is easy! Algorithm:

(1) Let $S := \{x_1\}$ and $i := 1$. [candidates $\mathcal{X} = \{x_1, \ldots, x_m\}$]

(2) While $i < m$, repeat:

- Let $i := i + 1$.
- If the majority graph restricted to $S \cup \{x_i\}$ is acyclic, then let $S := S \cup \{x_i\}$.

(3) Return the top element in $S$ (it is a Banks winner).

This algorithm has complexity $O(m^2)$ if given the majority graph, which in turn can be constructed in time $O(n \cdot m^2)$.

O. Hudry. A Note on "Banks Winners in Tournaments are Difficult to Recognize" by G.J. Woeginger. *Social Choice and Welfare*, 23(1):113–114, 2004.

# Summary

We have by now seen several types of for voting rules:

- *staged procedures:* STV, Plurality with Runoff, ...
- *positional scoring rules:* Borda, Plurality, Antiplurality, ...
- *Condorcet extensions:* Copeland, Banks, Kemeny, Young, ...

Helpful references for these and other voting rules are the works of Brams and Fishburn (2002) and Nurmi (1987).

We have also discussed three important properties:

- *Participation:* a voting rule should not give incentives not to vote (i.e., it should not suffer from the *no-show paradox*)
- *Condorcet principle:* elect the Condorcet winner whenever it exists
- *Complexity of winner determination:* computing the winner(s) of an election should be computationally tractable

S.J. Brams and P.C. Fishburn. Voting Procedures. In K.J. Arrow et al. (eds.), *Handbook of Social Choice and Welfare*, Elsevier, 2002.

H. Nurmi. *Comparing Voting Systems*. Kluwer Academic Publishers, 1987.

## What next?

In the next lecture we will see three different approaches to providing *characterisations* of voting rules.

- This will provide some explanation for the enormous diversity of voting rules encountered today.
- It will also connect to the impossibility theorems we have seen before, which may be considered characterisations of dictatorships.