Computational Social Choice 2021

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Plan for Today

We first are going to review a few more voting rules and remark on some surprising shortcomings of what look like reasonable rules.

To help us choose a good voting rule, we then discuss an approach to *characterising* rules using the so-called *axiomatic method*.

For full details see Zwicker (2016).

W.S. Zwicker. Introduction to the Theory of Voting. In F. Brandt et al. (eds.), *Handbook of Computational Social Choice*. CUP, 2016.

Preview: Some Axioms

We are going to use these *axioms* to highlight certain shortcomings of some of the voting rules we have seen and are going to see:

- *Participation Principle:* It should be in the best interest of voters to participate; voting truthfully should be no worse than abstaining.
- *Pareto Principle:* There should be no alternative that every voter strictly prefers to the alternative selected by the voting rule.
- *Condorcet Principle:* If there is an alternative that is preferred to every other alternative by a majority of voters, then it should win.

Reminder: The Model

Fix a finite set $A = \{a, b, c, \ldots\}$ of *alternatives*, with $|A| = m \geqslant 2$.

Let $\mathcal{L}(A)$ denote the set of all strict linear orders R on A. We use elements of $\mathcal{L}(A)$ to model (true) *preferences* and (declared) *ballots*.

Each member i of a finite set $N = \{1, \ldots, n\}$ of *voters* supplies us with a ballot R_i, giving rise to a *profile* $\boldsymbol{R} = (R_1, \ldots, R_n) \in \mathcal{L}(A)^n$.

A *voting rule* (or *social choice function*) for N and A selects (ideally) one or (in case of a tie) more winners for every such profile:

$$F : \mathcal{L}(A)^n \to 2^A \setminus \{\emptyset\}$$

If $|F(\boldsymbol{R})| = 1$ for all profiles $\boldsymbol{R}$, then F is called *resolute*.

Reminder: Some Voting Rules

So far we saw the following voting rules:

- Positional scoring rules: Borda, plurality, antiplurality, k-approval
- Based on majority graph: Copeland, Slater
- Based on weighted majority graph: Kemeny, ranked-pairs, (Borda)
- Plurality with runoff (generalisation to follow)

Runoff Methods: Single Transferable Vote & Variants

STV (used, e.g., in Australia) works in stages:

- If some alternative is top for an *absolute majority*, then it wins.
- Otherwise, the alternative ranked at the top by the fewest voters (the plurality loser) gets *eliminated* from the race.
- Votes for eliminated alternatives get *transferred:* delete removed alternatives from ballots and 'shift' rankings (i.e., if your 1st choice got eliminated, then your 2nd choice becomes 1st).

Various options for how to deal with *ties* during elimination.

In practice, voters need not be required to rank all alternatives (non-ranked alternatives are assumed to be ranked lowest).

For three alternatives, STV and *plurality with runoff* coincide.

Variants: *Coombs*, *Baldwin*, *Nanson* (different elimination criteria)

The No-Show Paradox

Under plurality with runoff (and thus under STV), it may be better to abstain than to participate and vote for your favourite alternative!

25 voters: $a \succ b \succ c$
46 voters: $c \succ a \succ b$
24 voters: $b \succ c \succ a$

Given these voter preferences, b gets eliminated in the first round, and c beats a 70:25 in the runoff.

Now suppose two voters from the first group abstain:

23 voters: $a \succ b \succ c$
46 voters: $c \succ a \succ b$
24 voters: $b \succ c \succ a$

Now a gets eliminated, and b beats c 47:46 in the runoff.

P.C. Fishburn and S.J Brams. Paradoxes of Preferential Voting. *Mathematics Magazine*, 1983.

Cup Rules via Voting Trees

We can define a voting rule via a *binary tree*, with the alternatives labelling the leaves, and an alternative progressing to a parent node if it beats its sibling in a *majority contest*.

Two examples for such *cup rules* and a possible profile of ballots:

```
(1)

        o
       / \
      o   c
     / \
    a   b
```

```
(2)     o
       / \
      /   \
     o     o
    / \   / \
   a   b b   c
```

$a \succ b \succ c$

$b \succ c \succ a$

$c \succ a \succ b$

Rule (1): c wins

Rule (2): a wins

Cup Rules and the Pareto Principle

The (weak) *Pareto Principle* requires that we should never elect an alternative that is strictly dominated in *every* voter's ballot.

Cup rules do not always satisfy this most basic of principles!

```
        o
       / \
      o   d
     / \
    o   a
   / \
  b   c
```

Consider this profile with three voters:

Ann:	$\mathtt{a} \succ \mathtt{b} \succ \mathtt{c} \succ \mathtt{d}$
Bob:	$\mathtt{b} \succ \mathtt{c} \succ \mathtt{d} \succ \mathtt{a}$
Cindy:	$\mathtt{c} \succ \mathtt{d} \succ \mathtt{a} \succ \mathtt{b}$

$\mathtt{d}$ *wins!* (despite being dominated by $\mathtt{c}$)

What happened? Note how this 'embeds' the Condorcet Paradox, with every occurrence of $\mathtt{c}$ being replaced by $\mathtt{c} \succ \mathtt{d}$ ...

Condorcet Extensions

An alternative that beats every other alternative in pairwise majority contests is called a *Condorcet winner*. Sometimes there is no CW.

The *Condorcet Principle* says that, if it exists, only the CW should win. Voting rules that satisfy this principle are called *Condorcet extensions*.

Exercise: *Show that Copeland, Slater, Kemeny, and cup rules are CEs.*

Two further Condorcet extensions:

- *Young:* Elect alternative x that minimises the number of voters we need to remove before x becomes the Condorcet winner.
- *Dodgson:* Elect alternative x that minimises the number of swaps of adjacent alternatives in the profile we need to perform before x becomes the Condorcet winner. (Note difference to Kemeny!)

Trivia: Dodgson is also known as Lewis Carroll (*Alice in Wonderland*).

Positional Scoring Rules and the Condorcet Principle

Consider this example with three alternatives and seven voters:

$$\begin{array}{ll} \text{3 voters:} & a \succ b \succ c \\ \text{2 voters:} & b \succ c \succ a \\ \text{1 voter:} & b \succ a \succ c \\ \text{1 voter:} & c \succ a \succ b \end{array}$$

So a is the *Condorcet winner:* a beats both b and c (with 4 out of 7). But any *positional scoring rule* makes b win (because $s_1 \geqslant s_2 \geqslant s_3$):

$$\begin{array}{ll} a: & 3 \cdot s_1 + 2 \cdot s_2 + 2 \cdot s_3 \\ b: & 3 \cdot s_1 + 3 \cdot s_2 + 1 \cdot s_3 \\ c: & 1 \cdot s_1 + 2 \cdot s_2 + 4 \cdot s_3 \end{array}$$

Thus, *no positional scoring rule* for three (or more) alternatives can possibly satisfy the *Condorcet Principle*.

Fishburn's Classification

Can classify voting rules on the basis of the *information* they require. The best known such classification is due to Fishburn (1977):

- *C1:* Winners can be computed from the *majority graph* alone.

  Examples: Copeland, Slater

- *C2:* Winners can be computed from the *weighted majority graph* (but not from the majority graph alone).

  Examples: Kemeny, ranked-pairs, Borda

- *C3:* All other voting rules. Examples: Young, Dodgson, STV

Remark: Fishburn originally intended this for Condorcet extensions only, but the concept also applies to all other voting rules.

P.C. Fishburn. Condorcet Social Choice Functions. *SIAM Journal on Applied Mathematics*, 1977.

Nonstandard Ballots

We defined voting rules over profiles of strict linear orders (even if some rules, e.g., plurality, don't use all information). Other options:

- *Approval voting:* You can approve of any subset of the alternatives. The alternative with the most approvals wins.
- *Even-and-equal cumulative voting:* You vote as for AV, but 1 point gets split evenly amongst the alternatives you approve.
- *Range voting:* You vote by dividing 100 points amongst the alternatives as you see fit (as long every share is an integer).
- *Majority judgment:* You award a grade to each of the alternatives ('excellent', 'good', etc.). Highest median grade wins.

The most important of these is approval voting.

Remark: *k-approval* and *approval voting* are very different rules!

Characterisation via Axiomatic Method

So many different voting rules! *How do you choose?*

One approach is to use the *axiomatic method* to identify voting rules of *normative* appeal. We will see one example for a classical result.

Axioms: Anonymity and Neutrality

Two basic fairness requirements for a voting rule F:

- F is *anonymous* if $F(R_1, \ldots, R_n) = F(R_{\pi(1)}, \ldots, R_{\pi(n)})$ for any profile $(R_1, \ldots, R_n)$ and any permutation $\pi : N \to N$.
- F is *neutral* if $F(\pi(\boldsymbol{R})) = \pi(F(\boldsymbol{R}))$ for any profile $\boldsymbol{R}$ and any permutation $\pi : A \to A$ (with π extended to profiles and sets of alternatives in the natural manner).

In other words:

- Anonymity is symmetry w.r.t. voters.
- Neutrality is symmetry w.r.t. alternatives.

Consequences of Axioms

For this slide only, let us restrict attention to voting rules for scenarios with just *two voters* $(n = 2)$ and *two alternatives* $(m = 2)$.

Exercise: *Show that there exists no resolute voting rule that is 'fair' in the sense of being both anonymous and neutral.*

Exercise: *But there still are a couple of irresolute voting rules that are both anonymous and neutral. Give some examples!*

Axiom: Positive Responsiveness

Notation: Write $N^{\boldsymbol{R}}_{x \succ y} = \{i \in N \mid (x, y) \in R_i\}$ for the set of voters who rank alternative x above alternative y in profile $\boldsymbol{R}$.

A (not necessarily resolute) voting rule satisfies *positive responsiveness* if, whenever some voter raises a (possibly tied) winner $x^\star$ in her ballot, then $x^\star$ will become the *unique* winner. Formally:

> F is *positively responsive* if $x^\star \in F(\boldsymbol{R})$ implies $\{x^\star\} = F(\boldsymbol{R'})$ for any alternative $x^\star$ and any two *distinct* profiles $\boldsymbol{R}$ and $\boldsymbol{R'}$ s.t. $N^{\boldsymbol{R}}_{x^\star \succ y} \subseteq N^{\boldsymbol{R}'}_{x^\star \succ y}$ and $N^{\boldsymbol{R}}_{y \succ z} = N^{\boldsymbol{R'}}_{y \succ z}$ for all $y, z \in A \setminus \{x^\star\}$.

Thus, this is a monotonicity requirement (we'll see others later on).

May's Theorem

When there are only *two alternatives*, then all the voting rules we have seen coincide with the *simple majority rule*. Good news:

Theorem 1 (May, 1952) *A voting rule for two alternatives satisfies anonymity, neutrality, and positive responsiveness if and only if that rule is the simple majority rule.*

This provides a good justification for using this rule (arguing in favour of 'majority' directly is harder than arguing for anonymity etc.).

K.O. May. A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decisions. *Econometrica*, 1952.

Proof Sketch

Clearly, the simple majority rule satisfies all three properties. ✓

Now for the other direction:

Assume the number of voters is *odd* $\rightsquigarrow$ no ties. (other case: similar)

There are two possible ballots: $a \succ b$ and $b \succ a$.
Anonymity $\rightsquigarrow$ only *number of ballots* of each type matters.

Consider all possible profiles $\boldsymbol{R}$. Distinguish two cases:

- Whenever $|N^{\boldsymbol{R}}_{a \succ b}| = |N^{\boldsymbol{R}}_{b \succ a}| + 1$, then only a wins.
  By *PR*, a wins whenever $|N^{\boldsymbol{R}}_{a \succ b}| > |N^{\boldsymbol{R}}_{b \succ a}|$. By *neutrality*, b wins otherwise. But this is just what the simple majority rule does. ✓

- There exist a profile $\boldsymbol{R}$ with $|N^{\boldsymbol{R}}_{a \succ b}| = |N^{\boldsymbol{R}}_{b \succ a}| + 1$, yet b wins.
  Suppose one a-voter switches to b, yielding $\boldsymbol{R'}$. By *PR*, now only b wins. But now $|N^{\boldsymbol{R'}}_{b \succ a}| = |N^{\boldsymbol{R'}}_{a \succ b}| + 1$, which is symmetric to the earlier situation, so by *neutrality* a should win. Contradiction. ✓

Young's Theorem

Another seminal result (which we won't discuss in detail here) is known as *Young's Theorem*. It provides a characterisation of the *PSR*'s.

It involves an axiom we have not yet seen:

> F satisfies *reinforcement* if, whenever we split the electorate into two groups and some alternative wins for both groups, then that alternative also wins for the full electorate:
>
> $$F(\boldsymbol{R}) \cap F(\boldsymbol{R'}) \neq \emptyset \;\;\Rightarrow\;\; F(\boldsymbol{R} \oplus \boldsymbol{R'}) = F(\boldsymbol{R}) \cap F(\boldsymbol{R'})$$

Young showed that a rule F is a *positional scoring rule* (with a scoring vector that need not be decreasing) iff it satisfies *anonymity*, *neutrality*, *reinforcement*, and a technical condition known as *continuity*.

H.P. Young. Social Choice Scoring Functions. *SIAM Journal on Applied Mathematics*, 28(4):824–838, 1975.

Other Approaches to Classifying Voting Rules

Attractive axiomatisations of several other voting rules exist as well. But there are also other approaches we can take to classify rules:

- Informational requirements ($\hookrightarrow$ Fishburn's classification)
- Computational requirements. Examples:
  - Borda: clearly tractable (straightforward polynomial algorithm)
  - STV: complexity seems to depend on how we break ties
  - Dodgson: looks highly intractable (and it is!)
- Distance-based rationalisation of voting rules. Examples:
  - Dodgson: return CW in (swap-distance)-closest profile with CW
  - Borda: return unanimous winner in closest profile with UW
- Epistemic characterisation: voting as truth-tracking ($\hookrightarrow$ later)

Summary

We have by now seen a very large number of voting rules:

- they explore different *intuitions* about how voting 'should' work and they seem to sometimes suffer from *counterintuitive problems*
- they differ in view of the *profile information* they require
- they differ in view of their *computational requirements*

We then saw an example for how to *characterise* a voting rule as the only rule that satisfies certain *axioms: May's Theorem*.

What next? More applications of the axiomatic method.