Homework #3

Deadline: Tuesday, 19 November 2024, 19:00

Exercise 1 (10 points)

Recall how we used a SAT solver to automatically prove the Gibbard-Satterthwaite Theorem for the special case of $n = 2$ voters and $m = 3$ alternatives. The purpose of this exercise is to explore some further applications of our implementation.

For $n = 2$ and $m = 3$, how many different resolute voting rules are there that are strategyproof? Answer this question by building on the code presented in class. Then provide a clear description and a suitable classification of these rules. (For instance, some of them will be dictatorships.) For this second part of the exercise, you may either extend our code further or you may resort to purely theoretical means.

Exercise 2 (10 points)

Recall that the Duggan-Schwartz Theorem establishes the impossibility of designing a (possibly irresolute) voting rule that, simultaneously, is (*i*) nonimposed, (*ii*) immune to manipulation by optimistic voters, (*iii*) immune to manipulation by pessimistic voters, and (*iv*) strongly nondictatorial. Prove it for the special case of $n = 2$ voters and $m = 3$ alternatives using the SAT technique. To help us understand your solution, for every axiom you implement, please report the number of clauses this axiom corresponds to.

Reuse anything you find helpful from the code presented in class (but clearly indicate which code you have copied and whether you have altered that code or left it unchanged).

Hints: This is a difficult exercise, but modelling the requirement of being strongly nondictatorial is relatively straightforward. So start with that. Modelling the two strategyproofness axioms requires some care, but you should end up with a fairly simple implementation as well. The main challenge is modelling nonimposition, which most immediately corresponds to a conjunction of disjunctions of conjunctions of literals. Translating this into CNF is impractical: the resulting formula would be huge (a conjunction of almost half a quintillion clauses of length 36). But you can use this trick: Introduce auxiliary variables $q_{r,x}$ with the intended meaning that in profile r alternative x is the *only* winner. Then express nonimposition with the help of these auxiliary variables, and fix their meaning by adding clauses that enforce $q_{r,x} \leftrightarrow p_{r,x} \wedge \neg p_{r,y} \wedge \neg p_{r,z}$ for all profiles r and (distinct) alternatives x, y, and z.

Besides proving the theorem, also demonstrate that for each of the four axioms featuring in the theorem it is possible to design a voting rule that satisfies the other three axioms (again, for the special case of $n = 2$ and $m = 3$). Report *how many* such voting rules there are for each of those four cases. Keep in mind that this corresponds to very demanding queries for the SAT solver, so you may not be able to obtain an answer in a reasonable amount of time. If one of the relevant queries does not return an answer within 5 minutes, please simply report this timeout instead of the relevant number of voting rules.