Introduction

Fair Division

Table of Contents

1. Fair Division
2. Fairness and Efficiency Criteria
3. The Problem
4. Tutorial Outline
5. Fair Division and Social Choice
6. Conclusion
Collective Utility Functions

A collective utility function (CUF) is a function that maps utility vectors to the reals. It is a systematic approach to defining social preferences.

\[ \text{CUF} : \mathbb{R}^n \rightarrow \mathbb{R} \]

SWO (Social Welfare Ordering)

Let \( A \) be a set of outcomes. An SWO \( R \) is a binary relation over \( A \) that is reflexive, transitive, and complete.

\[ \forall a, b, c \in A : \]
\[ (a, b) \in R \land (b, c) \in R \implies (a, c) \in R \]
\[ \forall a \in A : (a, a) \in R \]

Social Welfare

The idea goes back to Vilfredo Pareto (Italian economist, 1848–1923). Pareto efficiency is very often considered a minimum requirement for social welfare.

\[ u_i(a) \geq u_i(b) \quad \forall i \in [n] \quad \Rightarrow \quad a \succeq b \quad \forall a, b \in A \]

We need a systematic approach to defining social preferences. This is a reasonable definition, but it does not capture everything. A more refined approach is needed.

Collective Preference

By economy tradition (1759-1832), the minimum requirement for social welfare is to optimise overall profit. What is the allocation that allows the agents to solve the problem auctioneer has generated sufficient revenue?

\[ \max \sum_{i=1}^{n} u_i(a) \quad \forall a \in A \]

Maximising this function amounts to maximising the situation of the weakest member of society.

\[ \max \sum_{i=1}^{n} u_i(a) \quad \forall a \in A \]

What is the auctioneer's maximisation function?

\[ \max \sum_{i=1}^{n} u_i(a) \quad \forall a \in A \]

Pareto Efficiency

One approach to social welfare is to try to maximise overall profit. The idea goes back to Vilfredo Pareto (Italian economist, 1848–1923).

\[ u_i(a) \geq u_i(b) \quad \forall i \in [n] \quad \Rightarrow \quad a \succeq b \quad \forall a, b \in A \]

The weakest member of society has preferences over the allocation.

Egalitarian Social Welfare

The egalitarian variant of welfare economics is inspired by the work of John Rawls (American philosopher, 1921–2002) and has been formally developed, amongst others, by Amartya Sen since the 1970s (Nobel Prize in Economic Sciences in 1998).

\[ \min \sum_{i=1}^{n} u_i(a) \quad \forall a \in A \]

Maximising this function amounts to improving the situation of the weakest member of society.

\[ \min \sum_{i=1}^{n} u_i(a) \quad \forall a \in A \]

Each agent \( i \) has a valuation function \( v_i : A \rightarrow \mathbb{R} \).

\[ v_i(a) \geq v_i(b) \quad \forall a, b \in A_i \quad \Rightarrow \quad a_i \succeq b_i \quad \forall a, b \in A_i \]

Each agent's valuation function \( v_i \) is reflexive, transitive, and complete.

\[ \forall a, b \in A_i : v_i(a) \geq v_i(b) \quad \Rightarrow \quad a_i \succeq b_i \quad \forall a, b \in A_i \]

Social welfare ordering (SWO) can be defined in terms of preferences.

\[ u_i(a) \geq u_i(b) \quad \forall i \in [n] \quad \Rightarrow \quad a \succeq b \quad \forall a, b \in A \]

Maximising this function amounts to maximising the allocation's social preference.

\[ \max \sum_{i=1}^{n} u_i(a) \quad \forall a \in A \]

Social welfare ordering (SWO) can be defined in terms of preferences.
The veil of ignorance for multiagent systems

The SWO induced by the utilitarian CUF is zero if

\[ k \preceq \text{worthwhile to investigate egalitarian (and other)} \]

CUF measuring social welfare in

\[ \text{median-rank-dictator \quad elitist:} \]

Let

\[ \text{CUF for ordered utility vector} \]

Example:

\[ u \]

That means:

\[ \text{If you were to send a software agent into an artificial society to negotiate} \]

all

\[ u, v \]

Axiom 1 (PD)

A fair SWO will encourage inequality-reducing welfare redistributions.

The idea is due to Arthur C. Pigou (British economist, 1877-1959). The leximin ordering satisfies the Pigou-Dalton principle.

\[ \text{Be a utility vector.} \]

\[ \langle u \rangle \quad \text{lexically precedes} \quad \langle v \rangle \]

\[ \text{for all} \quad i, j \quad \text{such that:} \]

\[ i < k \quad \text{and} \quad j \in N \setminus \{i, k\} \]

\[ \exists \text{ a } \quad \text{lex-minority} \quad \text{leximin ordering} \quad \text{for} \quad i, j \]

interests are then of the following kind:

\[ \text{Interesting special cases:} \]

\[ \text{The Pigou-Dalton Principle} \]

新形势 is defined as the vector in increasing order.

\[ \text{Remark:} \]

\[ \text{For } u, v, w \in N \]

\[ \text{independent, while the egalitarian SWO is not.} \]

See Moulin (1988) for a precise statement of this result.

\[ \text{A given combination of axioms may be} \]

\[ \text{impossible} \]

\[ \text{satisfying a given (combination of) axiom(s).} \]

\[ \text{A given SWO may or may not satisfy a given axiom.} \]

\[ \text{For } k \in N \}

\[ \text{of the k-poorest agent:} \]

\[ \text{The k-rank dictator.} \]

\[ \text{CEU} \quad \text{aligned utility vectors.} \]

\[ \text{ordered utility vectors.} \]

\[ \text{Utilitarianism versus Egalitarianism} \]

\[ \text{in philosophy, economics, political science not.} \]

\[ \text{In fact, an SWO satisfies ZI iff it is represented by the utilitarian CUF.} \]

\[ \text{For } k \in N \]

\[ \text{the change is mean-preserving; and} \]

\[ \text{only} \]

\[ \text{for all } j \quad \text{and} \quad i \]

\[ \text{for all } n \]

\[ \text{in increasing order.} \]

\[ \text{The Nash Product} \]

\[ \prod \]

\[ \text{The Nash CUF is defined as the product of ordinal utilities:} \]

\[ \text{Nash Product} \]
Let $f$ be any SWO. If proportional utilities exist, clearly neither the utilitarian nor the egalitarian SWO are envy-free of an allocation. Envy-free allocations do not always exist if no agent would rather have one of the allocations. But what does that actually mean? The utilitarian SWO is not independent of the common utility pace, but the egalitarian SWO is. Any additive scale independent function of tax members of society will have to pay.

Example: $u_i$ is the utility given to the full bundle by agent $i$. For an SWO satisfying ICP only interpersonal comparisons (utilitarian, Nash, egalitarian and other rank dictators), the Pareto principle is satisfied.

Finally, if utility functions are independent of the common utility pace, but no the (cardinal) intensity of utility. We can therefore state that the SWO $\mathcal{S}$ is...
In this part of the tutorial, we are going to focus on cake-cutting and fair division. We want to divide a single divisible good, commonly referred to as a cake. The cake is represented by the unit interval $[0,1]$.

Fact 1

Proportionality and Envy-Freeness

- Each player is guaranteed at least one half
- No player will envy (any of) the other(s).

Is it possible to guarantee a fair division?

Beyond fairness, we may also be interested in the "operational" properties of the procedures themselves:

- Does the procedure guarantee that each player receives a single contiguous piece?
- Is the procedure deterministic or randomized?
- Does the procedure require an active referee?
- Is the procedure a discrete procedure that does not require a referee?
- Is the procedure easy to implement as a discrete sequence of queries to the agents?
- Is the procedure computationally tractable?

Cake Cutting Procedures

- Cut-and-choose cut-and-choose procedures are the simplest and the most intuitive way to divide a divisible good. They are also the simplest to implement and do not require a referee.
- For $n$ players, we can divide the cake into $n$ pieces: each player passes around a knife or a piece of the cake until all players have chosen a piece.
- Each player chooses a piece (in that order) and we are done.
- The resulting pieces do not have to be contiguous (namely if both players label two of them as "bad". — If player 2 passed, then players 3, 2, 1 play cut-and-choose. — If player 3 passed, then players 2, 3, 1 play cut-and-choose.
- Those methods are quite simple, but they are not envy-free in general.

The Steinhaus Procedure

Steinhaus proposed a discrete method of dividing a cake among three players:

1. Player 1 cuts the cake into three pieces (which she values equally).
2. That piece is passed around the players. Each player either lets it pass or can all actions be arbritary (as opposed to the minimum of 2 cuts).
3. If player 2 did not pass, then player 3 can also choose between passing or keeping the piece.
4. If neither player 2 or player 3 passed, then player 1 has to take (one of) the piece(s) labelled as "bad" by both 2 and 3. — The rest is reassembled and 2 and 3 play cut-and-choose.

Does this method guarantee a fair division (in the sense of envy-freeness)?

The Banach-Knaster Last-Diminisher Procedure

The Banach-Knaster Last-Diminisher Procedure

- The first player (if she is risk-averse) will choose the smaller piece.
- Any envy-free division is also proportional, but there are finite unions of subintervals of $[0,1]$, i.e., can it be minimal? If not, is it at least bounded?

The Banach-Knaster Last-Diminisher Procedure

- For $n > 2$ agents (called players), proportional divisions that are not envy-free are possible.
- Fact 2

Proportionality and Envy-Freeness

- The Steinhaus procedure — an envy-free cake-cutting algorithm — guarantees a fair division if the agents are risk-neutral. It is a discrete procedure that does not require a referee.
- The Banach-Knaster Last-Diminisher Procedure is continuous: the Intermediate-Value Theorem applies and something off (the "last diminisher") is obliged to take it.
- Each player is guaranteed at least one half
- No player will envy (any of) the other(s).

• The cake-cutting algorithm may fail to guarantee envy-freeness.

For $n > 3$ agents, the Steinhaus procedure is not envy-free anymore (unlike for $n = 2$).

Over the next few slides, we are first going to focus on cake-cutting and fair division. We want to divide a single divisible good, commonly referred to as a cake. We want to do so in a way that is fair to all players involved.
In the sequel, we will mostly focus on divisible goods, as these capture the essence of the problem. For indivisible goods, one usually has to resort to procedure (but other notions, such as equitability, Pareto efficiency, proportionality, and strategy-proofness . . . are also of interest).

For divisible goods, a natural approach is to break the problem down into smaller instances. There are many open problems relating to finding procedures with "good" properties for larger numbers. For example, the problem becomes non-trivial for more than two players, and requires the help of an arbitrary number of cuts. Even and Paz (1984) investigated (supposed to cut somewhere around the middle of the cake). The Dubins-Spanier Procedure (Dubins and Spanier, 1961) and The Stromquist Procedure (Stromquist, 1980) found an envy-free procedure for (n players, until just one player is left (who takes the rest).


Remark: • We have discussed various procedures for fairly dividing a cake (but other notions, such as equitability, Pareto efficiency, proportionality, and strategy-proofness . . . are also of interest). For divisible goods, a natural approach is to break the problem down into smaller instances. There are many open problems relating to finding procedures with "good" properties for larger numbers. For example, the problem becomes non-trivial for more than two players, and requires the help of an arbitrary number of cuts. Even and Paz (1984) investigated (supposed to cut somewhere around the middle of the cake). The Dubins-Spanier Procedure (Dubins and Spanier, 1961) and The Stromquist Procedure (Stromquist, 1980) found an envy-free procedure for (n players, until just one player is left (who takes the rest).

The first player to call "stop" receives the piece to the left of the knife. At the same time, each player is moving her own knife so that it would cut the righthand piece in half (wrt. her own valuation). If neither of the other two players hold the middle knife, then the other one gets the piece at mark). • • For • For • For • For

Even and Paz (1984) investigated (supposed to cut somewhere around the middle of the cake). The Dubins-Spanier Procedure (Dubins and Spanier, 1961) and The Stromquist Procedure (Stromquist, 1980) found an envy-free procedure for (n players, until just one player is left (who takes the rest).

The first player to call "stop" receives the piece to the left of the knife. At the same time, each player is moving her own knife so that it would cut the righthand piece in half (wrt. her own valuation). If neither of the other two players hold the middle knife, then the other one gets the piece at mark). • • For • For • For • For
The choice of language affects both welfare, then the allocation problem is equivalent to the winner determination problem in combinatorial Domains: From AI to Social Choice.

For fair division of indivisible goods we say about the properties of these emerging allocations?

Finding an allocation with maximal social welfare is at most equal to any individual utility:

Example:

Allocation of Indivisible Goods

Before we look into the "how", here are some complexity results:

Allocation

Bidding languages for combinatorial auctions (OR/XOR)

Logic-based languages (weighted goals)

Program-based preference representation (straight-line programs)

CP-nets and CI-nets (for ordinal preferences)

utility

The XOR-language means: Agent

Allocating 10 goods to 5 agents means an allocation exists is NP-complete; utility

 envy-free

For maximising goods

goods

An algorithm using (mixed)

utility (1st price auction)

value function

is a pair of allocations (before/after).

A deal may come with a number of side payments to compensate some of the agents for a loss in valuation. A

price offered for

payment function

welfare, then the allocation problem is equivalent to the winner determination problem in combinatorial Domains: From AI to Social Choice.

The choice of language affects both welfare, then the allocation problem is equivalent to the winner determination problem in combinatorial Domains: From AI to Social Choice.

For fair division of indivisible goods we say about the properties of these emerging allocations?

Finding an allocation with maximal

social welfare is at most equal to any individual utility:

Example:

Allocation of Indivisible Goods

Before we look into the "how", here are some complexity results:

Allocation

Bidding languages for combinatorial auctions (OR/XOR)

Logic-based languages (weighted goals)

Program-based preference representation (straight-line programs)

CP-nets and CI-nets (for ordinal preferences)

utility

The XOR-language means: Agent

Allocating 10 goods to 5 agents means an allocation exists is NP-complete; utility

 envy-free

For maximising goods

goods

An algorithm using (mixed)

utility (1st price auction)

value function

is a pair of allocations (before/after).

A deal may come with a number of side payments to compensate some of the agents for a loss in valuation. A

price offered for

payment function

welfare, then the allocation problem is equivalent to the winner determination problem in combinatorial Domains: From AI to Social Choice.

For fair division of indivisible goods we say about the properties of these emerging allocations?
The social surplus can be divided amongst all agents by using, say, the following payment function: 
\[ \text{util}_i(\delta) = v_i(\delta_i) - v_i(A - \delta_i) \]
where the latter would gain (not individually rational).

The only possible deal would be to move the whole table to agent \( \delta \) and only accept deals that increase social welfare:

\[ A \text{rationally agent (who does not plan ahead) will only accept deals that improve its individual welfare:} \]

\[ A, A \text{are willing to trade one chair for one table if they value the chair more than the latter would gain (not individually rational).} \]

The graph shows how utilitarian social welfare (utilitarianism) develops as agents attempt to contract more and more deals (black) leading to an allocation with maximal utilitarian social welfare would be: 

\[ (\{bob\}^{\text{chair}}, \{bob\}^{\text{table}}) \]

The bad news is that outcomes that maximise utilitarian social welfare is not achievable. By moving only one item, the agents can get the value of a bundle by adding up the values of its elements. That is, in a modular domain there are no synergies between items; you can get the value of a bundle by adding up the values of its elements.

If all valuation functions are modular, then IR deals suffice to guarantee outcomes with local/individual rationality. But it could be that general bilateral deals (red) achieve the same goal in fewer steps:

\[ (\{bob\}^{\text{chair}}, \{bob\}^{\text{table}}) \]

\[ \text{The Local/Individual Perspective} \]

The graph generated using the MADRAS platform of Buisman et al. (2007) shows a network of agents (the \( x \)-axis) developing as agents proceed to contract more and more deals (red) leading to an allocation with maximal utilitarian social welfare (the \( y \)-axis) amongst themselves. The agents can get the value of a bundle by adding up the values of its elements. Consequently the agent’s value for their bundle needs to be calculated:

\[ \sum_{i \in N} v_i(A_i) = v_i(A) \]

\[ v_i(\delta_i) = \sum_{j \in N} v_j(\delta_j) \]

The good news is that outcomes that maximise utilitarian social welfare is not achievable. By moving only one item, the agents can get the value of a bundle by adding up the values of its elements. That is, in a modular domain there are no synergies between items; you can get the value of a bundle by adding up the values of its elements.

If all valuation functions are modular, then IR deals suffice to guarantee outcomes with local/individual rationality. But it could be that general bilateral deals (red) achieve the same goal in fewer steps:

\[ (\{bob\}^{\text{chair}}, \{bob\}^{\text{table}}) \]

\[ \text{The Local/Individual Perspective} \]

The graph generated using the MADRAS platform of Buisman et al. (2007) shows a network of agents (the \( x \)-axis) developing as agents proceed to contract more and more deals (red) leading to an allocation with maximal utilitarian social welfare (the \( y \)-axis) amongst themselves. The agents can get the value of a bundle by adding up the values of its elements. Consequently the agent’s value for their bundle needs to be calculated:

\[ \sum_{i \in N} v_i(A_i) = v_i(A) \]

\[ v_i(\delta_i) = \sum_{j \in N} v_j(\delta_j) \]
For any given fairness or efficiency criterion, we would like to know how to set up a negotiation framework so as to be able to guarantee convergence to a social optimum. Some existing work:

- Pareto efficient outcomes via rational deals without money
- Outcomes maximising the egalitarian or the Nash CUF via specifically engineered deal criteria
- Envy-free outcomes via IR deals with a fixed payment function, for supermodular valuations (also on social networks)


Summary: Allocating Indivisible Goods

We have seen that finding a fair/efficient allocation in case of indivisible goods gives rise to a combinatorial optimisation problem. Two approaches:

- Centralised: Give a complete specification of the problem to an optimisation algorithm (related to combinatorial auctions).
- Distributed: Try to get the agents to solve the problem. For certain fairness criteria and certain assumptions on agent behaviour, we can predict convergence to an optimal state.

Literature

Besides listing fairness and efficiency criteria (Part 1), the "MARA Survey" also gives an overview of allocation procedures for indivisible goods. (It also covers applications, preference languages, and complexity results.) We have largely neglected strategic (and have been brief on algorithmic) aspects, which are better developed in the combinatorial auction literature. The handbook edited by Cramton et al. (2006) is a good starting point.

To find out more about convergence in distributed negotiation you may start by consulting the JAIR 2006 paper cited below.


Conclusion

Fair division is an important and exciting area of research. In this tutorial we have covered three topics:

- Fairness and efficiency defined in terms of individual preferences
- Classical algorithms for the cake-cutting problem (divisible goods)
- Combinatorial optimisation and negotiation for indivisible goods

These slides and the lecture notes will remain available on the tutorial website, and more extensive material can be found on the website of my Amsterdam course on Computational Social Choice:
