Collective Utility Functions

A Collective Utility Function (CUF) is a function $u$ that assigns a real number to each utility vector $(u_1, \ldots, u_n)$ of $n$ agents. Formally, a CUF $u$ is defined as follows:

$$u(x) = \sum_{i=1}^{n} u_i(x_i)$$

where $x = (x_1, \ldots, x_n)$ is an allocation of goods, possibly coupled with monetary side payments. Often, we can define social preference structures directly over utility vectors, rather than over utility functions themselves. This is known as classical utilitarianism (advocated, amongst others, by Jeremy Bentham, British philosopher, 1748–1832).

We will give an introduction to MARA mechanisms for the allocation of indivisible goods, possibly coupled with monetary side payments. Often, we can define social preference structures directly over utility vectors, rather than over utility functions themselves. This is known as classical utilitarianism (advocated, amongst others, by Jeremy Bentham, British philosopher, 1748–1832).

Remark: We also use the following notation:

$$u_i(x_i) = u_i \sim \approx$$

Observe that maximising this function amounts to maximising the average utility:

$$\frac{1}{n} \sum_{i=1}^{n} u_i(x_i)$$

The Nash (like the utilitarian) CUF favours increases in $u_i(x_i)$ for all $i$ and decreases in $u_j(x_i)$ for any $j \neq i$. Intuitively, if $u_i(x_i) \succ u_j(x_j)$ for some $i$ and $j$, then the agreement $(x_1, \ldots, x_n)$ produces less overall utility, but also inequality reductions (2 < 4). One approach to social welfare is to try to maximise overall profit. We will give an introduction to MARA mechanisms for the allocation of indivisible goods, possibly coupled with monetary side payments. Often, we can define social preference structures directly over utility vectors, rather than over utility functions themselves. This is known as classical utilitarianism (advocated, amongst others, by Jeremy Bentham, British philosopher, 1748–1832).


The egalitarian CUF measures social welfare as follows:

$$sw(x) = \min_{i=1}^{n} u_i(x_i)$$

Egalitarian Social Welfare

Intuitively, if $u_i(x_i) \succ u_j(x_j)$ for some $i$ and $j$, then the agreement $(x_1, \ldots, x_n)$ favours increases in $u_j(x_i)$ for all $j$, and decreases in $u_i(x_i)$ for any $i$. Often, we can define social preference structures directly over utility vectors, rather than over utility functions themselves. This is known as classical utilitarianism (advocated, amongst others, by Jeremy Bentham, British philosopher, 1748–1832).

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Let $u$ be a given (class of) SWO(s) and $CUF$ be a given (combination of) axiom(s).

Axiom 1 (ZI)

An SWO $\bar{u}$ is zero independent if and only if $\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) = \bar{u}(\langle v_1, v_2, \ldots, v_n \rangle)$ for all $\langle u_1, u_2, \ldots, u_n \rangle \in R$ and $\langle v_1, v_2, \ldots, v_n \rangle \in R$.

Axiom 2 (SI)

An SWO $\bar{u}$ is scale independent if and only if $\bar{u}(\lambda \cdot \langle u_1, u_2, \ldots, u_n \rangle) = \lambda \cdot \bar{u}(\langle u_1, u_2, \ldots, u_n \rangle)$ for all $\lambda > 0$.

Axiom 3 (ICP)

An SWO $\bar{u}$ is independent of the common utility pace if and only if $\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) = \bar{u}(\langle u_1 + k, u_2 + k, \ldots, u_n + k \rangle)$ for all $k$.

Axiom 4 (ADD)

An SWO $\bar{u}$ is additive if and only if $\bar{u}(\langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle) = \bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) + \bar{u}(\langle v_1, v_2, \ldots, v_n \rangle)$ for all $v_1, v_2, \ldots, v_n$.

Axiom 5 (MON)

An SWO $\bar{u}$ is monotonic if and only if $\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) \geq \bar{u}(\langle v_1, v_2, \ldots, v_n \rangle)$ for all $u_1 \geq v_1, u_2 \geq v_2, \ldots, u_n \geq v_n$.

Axiom 6 (ENV)

An SWO $\bar{u}$ is envy-free if and only if $\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) > \bar{u}(\langle u_2, u_1, \ldots, u_n \rangle)$ for all $u_1 > u_2$.

Axiom 7 (PRO)

An SWO $\bar{u}$ is proportional if and only if $\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) = \frac{u_1}{u_1 + u_2 + \cdots + u_n}$ for all $u_1 > 0$.

Axiom 8 (Pareto)\textsuperscript{25}

An SWO $\bar{u}$ satisfies Pareto efficiency if and only if $\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) > \bar{u}(\langle v_1, v_2, \ldots, v_n \rangle)$ for all $u_i > v_i$.

Axiom 9 (Rank Dictators)\textsuperscript{26}

An SWO $\bar{u}$ is a rank dictator CUF if and only if $\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) = \bar{u}(\langle u_{\text{rank}}(1), u_{\text{rank}}(2), \ldots, u_{\text{rank}}(n) \rangle)$ for all $\langle u_1, u_2, \ldots, u_n \rangle$.

Axiom 10 (Leximin)\textsuperscript{27}

An SWO $\bar{u}$ is leximin-ordering if and only if $\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) = \bar{u}(\langle u_{\text{leximin}(1)}, u_{\text{leximin}(2)}, \ldots, u_{\text{leximin}(n)} \rangle)$ for all $\langle u_1, u_2, \ldots, u_n \rangle$.

In fact, an SWO $\bar{u}$ satisfies ZI iff it is represented by the utilitarian CUF. See Moulin (1988) for a precise statement of this result.

We now introduce an SWO that may be regarded as a refinement of the SWO induced by the egalitarian CUF. Such an SWO is called the leximin CUF.

Let $k$ denote the number of agents.

Theorem 1 (existence of the leximin CUF)\textsuperscript{28}

There exists a SWO $\bar{u}$ such that $\bar{u}$ is leximin-ordering and zero independent.

Proof.\textsuperscript{29}

The SWO $\bar{u}$ is defined as follows:

$\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) = \bar{u}(\langle u_{\text{leximin}(1)}, u_{\text{leximin}(2)}, \ldots, u_{\text{leximin}(n)} \rangle)$ for all $\langle u_1, u_2, \ldots, u_n \rangle$.

Next we give a couple of examples for different agents and different SWOs.

Example 1: Utilitarian CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 20$.

Example 2: Egalitarian CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Example 3: Leximin CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Example 4: Rank Dictators CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Example 5: SI CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Example 6: ICP CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Example 7: ADD CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Example 8: MON CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Example 9: ENV CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Example 10: PRO CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Example 11: Pareto CUF

Let $u_1 = 5$, $u_2 = 20$, and $u_3 = 0$. We obtain an SWO $\bar{u}$ such that $\bar{u}(\langle u_1, u_2, u_3 \rangle) = 5$.

Theorem 2 (Existence of the leximin CUF)\textsuperscript{30}

There exists a SWO $\bar{u}$ such that $\bar{u}$ is leximin-ordering and zero independent.

Proof.\textsuperscript{31}

The SWO $\bar{u}$ is defined as follows:

$\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) = \bar{u}(\langle u_{\text{leximin}(1)}, u_{\text{leximin}(2)}, \ldots, u_{\text{leximin}(n)} \rangle)$ for all $\langle u_1, u_2, \ldots, u_n \rangle$.

Theorem 3 (Existence of the rank dictator CUF)\textsuperscript{32}

There exists a SWO $\bar{u}$ such that $\bar{u}$ is rank dictator CUF.

Proof.\textsuperscript{33}

The SWO $\bar{u}$ is defined as follows:

$\bar{u}(\langle u_1, u_2, \ldots, u_n \rangle) = \bar{u}(\langle u_{\text{rank}(1)}, u_{\text{rank}(2)}, \ldots, u_{\text{rank}(n)} \rangle)$ for all $\langle u_1, u_2, \ldots, u_n \rangle$.
We can distinguish two approaches: centralised and distributed.

The centralised approach utilises the global perspective to assess how well we are doing. We are not going to talk about designing a concrete negotiation protocol, but rather study the framework from an abstract point of view. Furthermore, suppose the initial allocation of goods is even, and say about the properties of these emerging allocations?

We can use the global perspective to assess how well we are doing. Observe that there is no need to include the agents' monetary valuation functions.

Next we will consider the problem of allocating indivisible goods.

Social welfare for allocation $A$ is maximising

$$\sum_{i \in A} v_i(\cap_r R)$$

Checking whether an allocation is NP-hard, even when all valuations are modular.

Finding an allocation with maximal social welfare is also coNP-complete.

If all valuations are modular, then it is polynomial. Social welfare is also polynomial if all valuations are a function of a single good.

Observe that there is a function $p$ such that improving individual welfare:

$$\sum_{i \in A} v_i(\cap_r R) \rightarrow p$$

A rational agent (who does not plan ahead) will only accept deals that strictly outweigh any loss in money.

There is a need to devise an optimisation algorithm to compute an allocation in a centralised manner, we now want agents to be able to do this in a distributed manner by contracting deals locally.

The Local/Individual Perspective

Example:

Example:

Let $\mathbb{A} = \{a, b, c\}$ be the allocation.

Each agent $i \in \mathbb{A}$ has got a payment function $\delta$ and a valuation function $v$.


delta = \{ (\text{chair}, 7), (\text{table}, 5) \}
v_i(\cup_r R) = 5 \quad \forall i \in \mathbb{A}

In the next slides, we will consider the following valuation functions:

\begin{align*}
  & a = \{ (\text{chair}, 7), (\text{table}, 5) \} \\
  & b = \{ (\text{chair}, 5), (\text{table}, 5) \} \\
  & c = \{ (\text{chair}, 5), (\text{table}, 7) \} \\
  & 0 = \{ (\text{chair}, 5), (\text{table}, 1) \}
\end{align*}

For the remainder of today we will work in this framework:

$$\bigcup_{a=\mathbb{A}} v(a) = 7$$

We are not going to talk about designing a concrete negotiation protocol, but rather study the framework from an abstract point of view.

In the next slides, we will consider the allocation $A$ for the following preferences:

\begin{align*}
  & a = \{ (\text{chair}, 7), (\text{table}, 5) \} \\
  & b = \{ (\text{chair}, 5), (\text{table}, 5) \} \\
  & c = \{ (\text{chair}, 5), (\text{table}, 7) \} \\
  & 0 = \{ (\text{chair}, 5), (\text{table}, 1) \}
\end{align*}

We are not going to talk about designing a concrete negotiation protocol, but rather study the framework from an abstract point of view.

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\end{align*}

We are not going to talk about designing a concrete negotiation protocol, but rather study the framework from an abstract point of view.
A deal $A$ is individually rational if $\delta(A, A') = 0$.

The lemma confirms that individually rational deals are exactly those deals with side payments that are optimal, i.e. there exists an allocation $A$ that is optimal, i.e. there exists an allocation $A$ such that

$$\delta(A, A) = 0.$$ 

Proof: Let $A$ be the terminal allocation. Assume there is a deal $A'$. Then, by our first lemma, $A'$ is individually rational if

$$\delta(A', A) = 0.$$ 

It is now easy to prove the following theorem.

Theorem 1 (Sandholm, 1998)

Any allocation that increases overall social welfare eventually result in an allocation with maximal social welfare.

Results

Contract Types for Satisficing Task Allocation: I Theoretical Foundations


Contract Types for Satisficing Task Allocation: II Theoretical Foundations


What if we know nothing?

More MARA


Summary

Linking the Local and the Global Perspectives