Summer School on Fair Division (FairDiv-2015): Tutorial on Cake Cutting

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Cake Cutting

We will discuss methods for dividing a single divisible (heterogeneous) resource (the "*cake*") between several agents.

Studied seriously since the 1940s (Banach, Knaster, Steinhaus).
Simple model, yet still many open problems.

Plan for this tutorial:

- Definition of the problem: how can you cut a cake fairly?
- Presentation of several protocols for cutting a cake
- Complexity analysis: how many cuts do you need?

S.J. Brams and A.D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press, 1996.

J. Robertson and W. Webb. *Cake-Cutting Algorithms*. A.K. Peters, 1998.

U. Endriss. *Lecture Notes on Fair Division*. ILLC, University of Amsterdam, 2009.

A.D. Procaccia. Cake Cutting Algorithms. In F. Brandt et al. (eds.), *Handbook of Computational Social Choice*. CUP, 2015. In press.

Formal Model

The *cake* is the interval $[0, 1]$ of the real numbers from 0 to 1:

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|----------------------|
0                      1
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We need to divide the cake between n *agents* (with $n = 2, 3, 4, 5, \ldots$).
A *piece* of cake is a finite union of disjoint subintervals of $[0, 1]$.

Each agent i has a *valuation function* v_i to measure how much she likes any given piece of cake. Assumptions:

- Normalisation: $v_i(\texttt{full\_cake}) = 1$ and $v_i(\texttt{nothing}) = 0$
- Additivity: $v_i(A \cup B) = v_i(A) + v_i(B)$ if A and B don't overlap
- Continuity: small increases in cake $\Rightarrow$ small increases in value

Proportional Fairness

We want to design protocols that are "fair". What does that mean?

One possible definition:

> *An allocation of pieces of cake to agents is proportionally fair, if every agent's subjective value for her piece is at least $\frac{1}{n}$.*

Other options: *envy-freeness* (discussed later), *equitability* (not today)

But more precisely, we want this:

> *A cake-cutting protocol is proportionally fair, if every agent can ensure she gets a piece that she values at at least at $\frac{1}{n}$.*

For all proportionally fair protocols we will see, agents can in fact guarantee their fair share by answering all questions *truthfully*.

So for agents who only care about their fair share, these protocols are *strategy-proof* (but *not* in the standard game-theoretical sense).

Cut-and-Choose Protocol

For the case of *2 agents*, you all know how to do this:

- *One agent cuts the cake in two pieces (of equal value to her), and the other chooses one of them (the piece she prefers).*

This clearly is *proportionally fair!*

Remark: Truthfully answering the questions ("where is the middle?" and "which one do you prefer?") is the best you can do. But if the cutter knows the valuation of the chooser, she can do even better (so this is *not* strategy-proof for a value-maximising agent).

What about 3 agents? Or more?

The Steinhaus Protocol

This *proportional* protocol for *three agents* was proposed by Steinhaus around 1943. Our exposition follows Brams and Taylor (1995).

(1) Agent 1 cuts the cake into three pieces (which she values equally).

(2) Agent 2 "passes" (if she thinks at least two of the pieces are $\geqslant 1/3$) or labels two of them as "bad". — If agent 2 passed, then agents 3, 2, 1 each choose a piece (in that order) and we are done. ✓

(3) If agent 2 did not pass, then agent 3 can also choose between passing and labelling. — If agent 3 passed, then agents 2, 3, 1 each choose a piece (in that order) and we are done. ✓

(4) If neither agent 2 or agent 3 passed, then agent 1 has to take (one of) the piece(s) labelled as "bad" by both 2 and 3. —
The rest is reassembled and 2 and 3 play cut-and-choose. ✓

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

The Dubins-Spanier Moving-Knife Protocol

Dubins and Spanier (1961) proposed this protocol (for any n):

(1) A referee moves a *knife* slowly across the cake, from left to right. Any agent may shout "*stop*" at any time. Whoever does so receives the piece to the left of the knife.

(2) When a piece has been cut off, we *continue* with the remaining agents, until just one agent is left (who takes the rest).

This is *proportionally fair!* (Of course: right-to-left works as well.)

Exercise 1: You *love* strawberries. There is a single large strawberry on the right end of the cake. Do you prefer left-to-right or right-to-left?

Exercise 2: How would you program a computer to play for you?

L.E. Dubins and E.H. Spanier. How to Cut a Cake Fairly. *American Mathematical Monthly*, 68(1):1–17, 1961.

Problem

Each agent has to continuously monitor the knife as it moves over all the real numbers from 0 to 1. For each number, the agent has to evaluate the piece to the left of the knife. This is impossible.

The Robertson-Webb Model

What counts as a "protocol"? — A reasonable protocol should be implementable in terms of just two types of queries:

- $\mathrm{Cut}_i(x, \alpha) \mapsto y$: Ask agent i to cut off a piece of value α, starting from point x (she cuts at point y).
- $\mathrm{Eval}_i(x, y) \mapsto \alpha$: Ask agent i to indicate her value for the piece between points x and y (she answers α).

Now we can count queries and compare the complexity of protocols.

J. Robertson and W. Webb. *Cake-Cutting Algorithms*. A.K. Peters, 1998.

Simulating the Moving-Knife Protocol

We can "discretise" the moving-knife protocol to solve our problem:

(1) Ask each agent to *mark* the cake where she *would* shout "stop". Then *cut* the cake at the *leftmost mark* and give the resulting piece to the agent who made that mark.

(2) When a piece has been cut off, we *continue* with the remaining agents, until just one agent is left (who takes the rest).

Formally, the marks are cut-queries. No evaluation-queries needed.

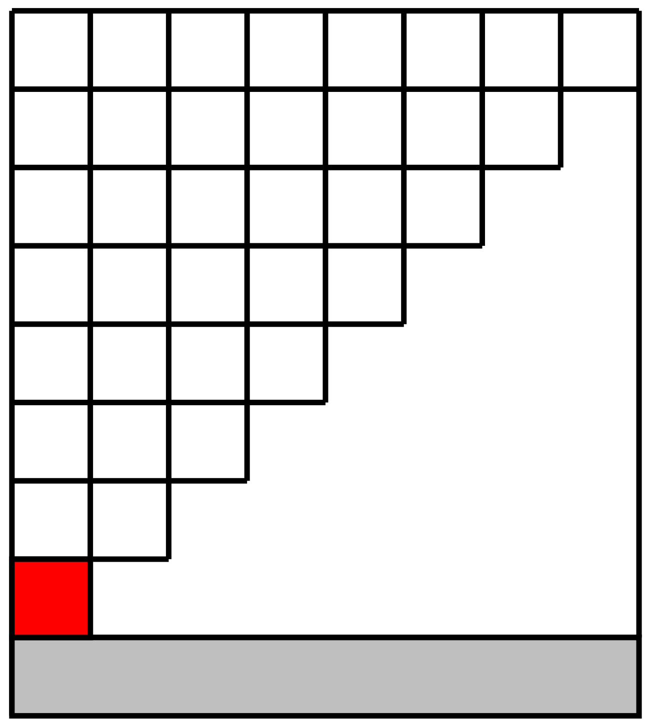
Exercise: How *complex* is this (*how many queries* do we need)?

Complexity Analysis: Number of Marks

In each round, each participating agent makes one mark. The number of participating agents goes down from n to 2. Thus:

$$n + (n-1) + (n-2) + \cdots + 3 + 2 \;=\; \frac{n \cdot (n+1)}{2} - 1 \;\approx\; \frac{1}{2} \cdot n^2$$

Proof:



Can we do better?

The Even-Paz Divide-and-Conquer Protocol

Even and Paz (1984) introduced the divide-and-conquer protocol:

(1) Ask each agent to put a *mark* on the cake.

(2) *Cut* the cake at the $\lfloor \frac{n}{2} \rfloor$*th mark* (counting from the left).

Associate the agents who made the *leftmost* $\lfloor \frac{n}{2} \rfloor$ *marks* with the *lefthand part*, and the *remaining agents* with the *righthand part*.

(3) *Repeat* for each group, until only one agent is left.

This also is *proportionally fair!* Again, we only require cut-queries.

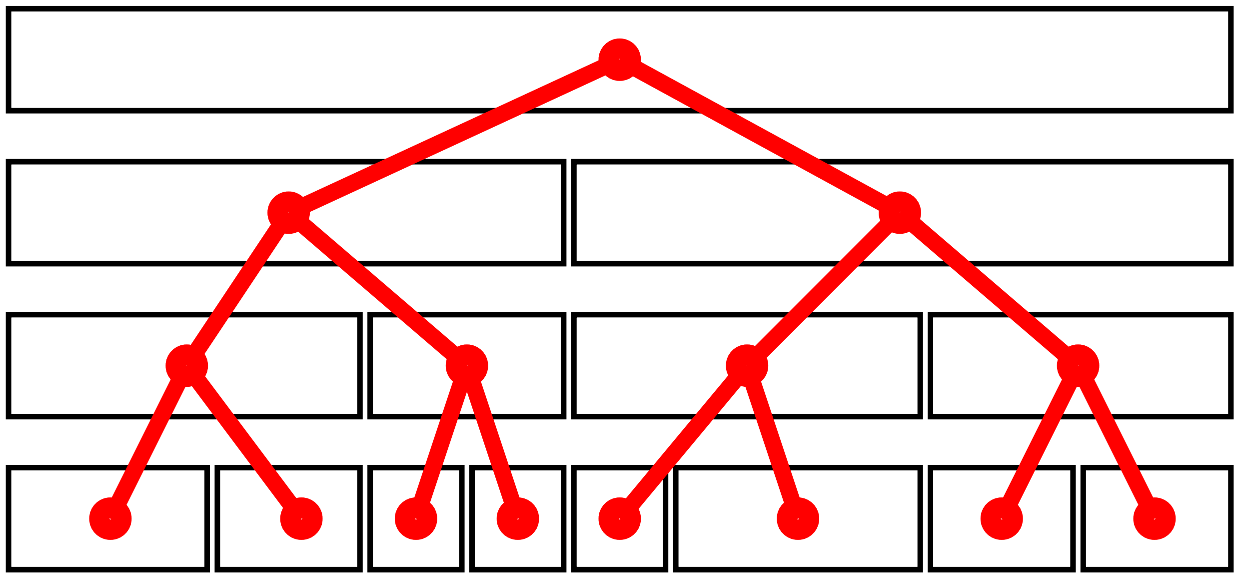
Exercise: How *complex* is this?

S. Even and A. Paz. A Note on Cake Cutting. *Discrete Applied Mathematics*, 7(3):285–296, 1984.

Complexity Analysis: Number of Marks

In each round, every agent makes one mark. So: *n marks per round*

But how many rounds?



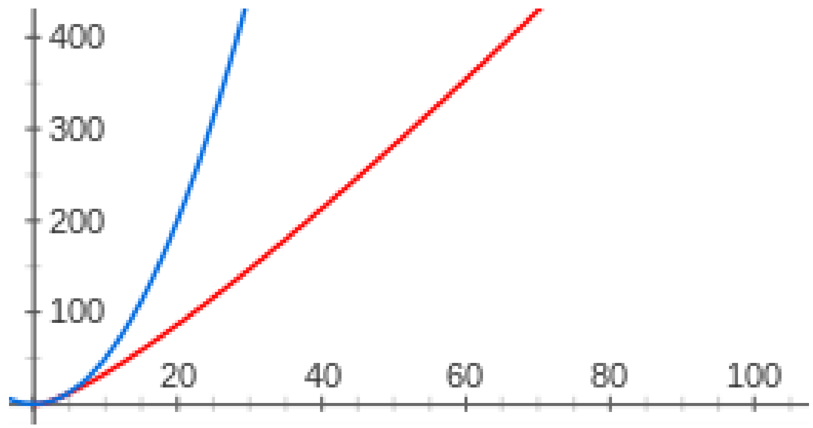
rounds $=$ number of times you can divide n by 2 before hitting $\leqslant 1$

$\approx \log_2 n$ (example: $\log_2 8 = 3$)

Thus: number of *marks* required $\approx n \cdot \log_2 n$

Comparison and Limitations

Recall: *simulated moving-knife* requires around $\frac{1}{2} \cdot n^2$ marks and *divide-and-conquer* requires around $n \cdot \log_2 n$ marks.



So: divide-and-conquer is much better (for large n, complexity-wise).

And in fact divide-and-conquer is the best you can do:

Theorem 1 (Edmonds and Pruhs, 2006) *Any proportionally fair protocol requires $\Omega(n \log n)$ queries in the Robertson-Webb model.*

J. Edmonds and K. Pruhs. Cake cutting really is not a piece of cake. SODA-2006.

Envy

Proportional fairness is but one formalisation of "fairness":

> *A cake-cutting protocol is called envy-free, if every agent can ensure that she will receive a subjectively largest piece.*

Connections between these two notions of fairness:

- Observe that for $n = 2$ *agents*, we have:

$$\textit{envy-freeness} \Longleftrightarrow \textit{proportional fairness}$$

- But for $n \geqslant 3$ *agents*, we only have:

$$\textit{envy-freeness} \Longrightarrow \textit{proportional fairness}$$

Indeed, of our protocols only *cut-and-choose* guarantees envy-freeness.

<u>Exercise:</u> Give an example where divide-and-conquer violates EF.

No fully satisfactory solution for envy-free cake cutting is known!

Four Simultaneously Moving Knives

Stromquist (1980) found this *envy-free* protocol for *3 agents:*

- A referee slowly moves a knife across the cake, from left to right (supposed to eventually cut somewhere around the $\frac{1}{3}$ mark).
- At the same time, each agent is moving her own knife so that it would cut the righthand piece in half (w.r.t. her own valuation).
- The first agent to call "stop" receives the piece to the left of the referee's knife. The righthand part is cut by the middle one of the three agent knifes. If neither of the other two agents hold the middle knife, they each obtain the piece at which their knife is pointing. If one of them does hold the middle knife, then the other one gets the piece at which her knife is pointing.

W. Stromquist. How to Cut a Cake Fairly. *American Mathematical Monthly*, 87(8):640–644, 1980.

The Selfridge-Conway Protocol

The first discrete protocol achieving *envy-freeness* for $n = 3$ agents has been discovered independently by Selfridge and Conway (around 1960). It doesn't ensure contiguous pieces. Our exposition follows Brams and Taylor (1995).

(1) Agent 1 cuts the cake in three pieces (she considers equal).

(2) Agent 2 either "passes" (if she thinks at least two pieces are tied for largest) or trims one piece (to get two tied for largest pieces). —
If she passed, then let agents 3, 2, 1 pick (in that order). ✓

(3) If agent 2 did trim, then let 3, 2, 1 pick (in that order), but require 2 to take the trimmed piece (unless 3 did). Keep the trimmings unallocated for now (note: the partial allocation is envy-free).

(4) Now divide the trimmings. Whoever of 2 and 3 received the *un*trimmed piece does the cutting. Let agents choose in this order: non-cutter, agent 1, cutter. ✓

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

Limitations and Open Problems

Achieving *envy-freeness* is much harder than achieving proportionality:

- For $n = 2$ the problem is easy: cut-and-choose does the job.
- For $n = 3$ we saw two protocols, each with some drawbacks.
- For $n = 4$, no protocol producing *contiguous pieces* is known.

For *arbitrary* n, Brams and Taylor (1995) give an envy-free protocol requiring an *unbounded* number of queries in the R-W model.

Theorem 2 (Procaccia, 2009) *Any envy-free protocol requires $\Omega(n^2)$ queries in the Robertson-Webb model.*

Open what the best bound is (must be between n^2 and *unbounded*). For comparison: proportionality only requires $O(n \log n)$ many queries.

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

A.D. Procaccia. Thou Shalt Covet Thy Neighbor's Cake. IJCAI-2009.

Last Slide

This has been an introduction to cake cutting. We've seen:

- usable protocols for guaranteeing *proportional fairness*
- severe limitations for protocols guaranteeing *envy-freeness*

In terms of methodology, we have discussed:

- how to define *fairness* in terms of guarantees for the agents
- how to formalise the concept of *"protocol"* (Robertson-Webb)
- how to analyse the *complexity* of a cake-cutting protocol

Topics not covered:

- better results for *restricted* types of *valuation functions*
- other *fairness properties*, including notions of *approximate fairness*
- *price of fairness* (see tutorial by Iannis Caragiannis)
- *randomised* protocols
- *game-theoretical* analysis
- ...