Game Theory 2025

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Plan for Today

The coming two lectures are about *cooperative game theory*, where we study so-called *coalitional games* and the *formation of coalitions*.

Specifically, these two lectures are about *transferable-utility games*. Today we focus on *stability* for such games:

- definition of transferable-utility games
- examples for transferable-utility games
- the core: set of surplus divisions that are stable

Part of this is also covered in Chapter 8 of the *Essentials*.

K. Leyton-Brown and Y. Shoham. *Essentials of Game Theory: A Concise, Multidisciplinary Introduction*. Morgan & Claypool Publishers, 2008. Chapter 8.

Coalitional Games

A *transferable-utility coalitional game* in characteristic-function form (or simply: a *TU game*) is a tuple $\langle N, v \rangle$, where

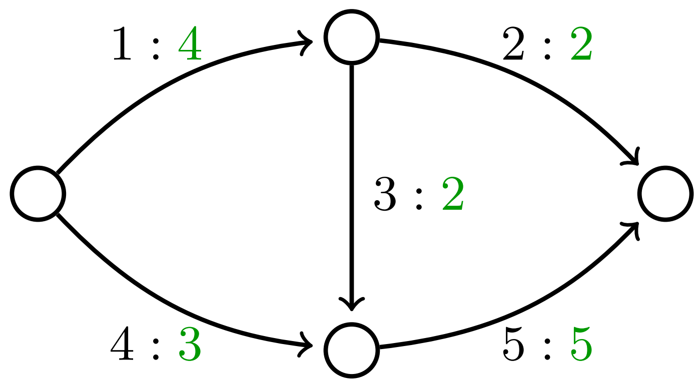
- $N = \{1, \ldots, n\}$ is a finite set of *players* and
- $v : 2^N \to \mathbb{R}_{\geqslant 0}$, with $v(\emptyset) = 0$, is a *characteristic function*, mapping every possible *coalition* $C \subseteq N$ to its *surplus* $v(C)$.

Note: The surplus $v(C)$ is also known as the *value* or the *worth* of C.

The players are assumed to form coalitions (thereby partitioning N). Each coalition C receives its surplus $v(C)$ and—*somehow*—divides it amongst its members (possible due to utility being transferable).

Example: A Network Flow Game

Each pipeline is owned by a different player $(1, 2, \ldots)$. Each pipeline is annotated as [*owner* : *capacity*]. The surplus $v(C)$ for coalition C is the amount of oil it can pump through the part of the network it owns.



We obtain $v(12) = 2$, $v(45) = 3$, $v(15) = 0$, $v(134) = 0$, $v(135) = 2$, $v(1345) = 5$, $v(12345) = 7$, and so forth.

Exercise: *What coalition(s) will form? How to divide the surplus?*

Simple Games

A *simple game* is a TU game $\langle N, v\rangle$ for which it is the case that $v(C) \in \{0,1\}$ for every possible coalition $C \subseteq N$, and $v(N) = 1$.

Thus: every coalition is either *winning* or *losing*.

Voting Games

A (weighted) *voting game* is a tuple $\langle N, \boldsymbol{w}, q\rangle$, where

- $N = \{1, \ldots, n\}$ is a finite set of *players*;
- $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n_{\geqslant 0}$ is a vector of *weights*; and
- $q \in \mathbb{R}_{>0}$ is a *quota* with $q \leqslant w_1 + \cdots + w_n$.

Coalition $C \subseteq N$ is *winning*, if the sum of the weights of its members meets or exceeds the quota. Otherwise it is *losing*.

Thus, a voting game $\langle N, \boldsymbol{w}, q\rangle$ is in fact a simple game $\langle N, v\rangle$ with:

$$v(C) \quad = \quad \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geqslant q \\ 0 & \text{otherwise} \end{cases}$$

<u>Exercise:</u> *Explain why we indeed get $v(N) = 1$, as required.*

Example: Council of the European Commission

In the *Treaty of Rome* (1957) the founding countries of the EU fixed the voting rule to be used in the Council of the European Commission:

- To pass, a proposal had to get at least 12 votes in favour.
- France, Germany, and Italy each had 4 votes. Belgium and the Netherlands each had 2 votes. Luxembourg had 1 vote.

This is a weighted voting game $\langle N, \boldsymbol{w}, q\rangle$ with

- $N = \{\text{BE}, \text{DE}, \text{FR}, \text{IT}, \text{NL}, \text{LU}\}$
- $w_{\text{DE}} = w_{\text{FR}} = w_{\text{IT}} = 4$, $w_{\text{BE}} = w_{\text{NL}} = 2$, and $w_{\text{LU}} = 1$
- $q = 12$

Exercise: *Is this fair? What about Luxembourg in particular?*

Properties of Coalitional Games

Some TU games $\langle N, v\rangle$ have certain properties (for all $C, C' \subseteq N$):

- *additive:* $C \cap C' = \emptyset$ implies $v(C \cup C') = v(C) + v(C')$
- *superadditive:* $C \cap C' = \emptyset$ implies $v(C \cup C') \geqslant v(C) + v(C')$
- *convex:* $v(C \cup C') \geqslant v(C) + v(C') - v(C \cap C')$
- *cohesive:* $N = C_1 \uplus \cdots \uplus C_K$ implies $v(N) \geqslant v(C_1) + \cdots + v(C_K)$
- *monotonic:* $C \subseteq C'$ implies $v(C) \leqslant v(C')$

Remark: Additive games are not interesting. No synergies between players: every coalition structure is equally good for everyone.

Exercise: *Show that convexity can equivalently be expressed as* $v(S' \cup \{i\}) - v(S') \geqslant v(S \cup \{i\}) - v(S)$ *for* $S \subseteq S' \subseteq N \setminus \{i\}$.

Exercise: *Show that additive $\Rightarrow$ convex $\Rightarrow$ superadditive $\Rightarrow$ cohesive, and also that superadditive $\Rightarrow$ monotonic.*

Examples

What are the properties of the special types of games we have seen?

- *Network flow games* are easily seen to be *monotonic* as well as *superadditive*, but they usually are not *convex*.
- *Voting games* are *monotonic*, but *not* necessarily *cohesive*.

Issues

The central questions in coalitional game theory are:

- *Which coalitions will form?*
- *How should the members of coalition C divide their surplus $v(C)$?*
  - *What would be a division that ensures stability?* (this lecture)
  - *What division would be fair?* (next lecture)

Often, the forming of the *grand coalition* N is considered the goal.
This is particularly reasonable for games that are superadditive.

Example

Consider the following 3-player TU game $\langle N, v \rangle$, with $N = \{1,2,3\}$, in which no single player can generate any surplus on her own:

$$\begin{array}{lll} v(\{1\}) = 0 & v(\{1,2\}) = 7 & v(N) = 10 \\ v(\{2\}) = 0 & v(\{1,3\}) = 6 & \\ v(\{3\}) = 0 & v(\{2,3\}) = 5 & \end{array}$$

Exercise: *What coalition(s) will form? How to divide the surplus?*

Payoff Vectors and Imputations

Suppose the grand coalition has formed. *How to divide its surplus?*

Recall: $v(N)$ is the surplus of the grand coalition N, and $n = |N|$.

A *payoff vector* is a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geqslant 0}$. Properties:

- $\boldsymbol{x}$ is *feasible* if $\sum_{i \in N} x_i \leqslant v(N)$: do no allocate more than there is
- $\boldsymbol{x}$ is *efficient* if $\sum_{i \in N} x_i = v(N)$: allocate all there is
- $\boldsymbol{x}$ is *individually rational* if $x_i \geqslant v(\{i\})$ for all players $i \in N$: nobody should be able to do better on their own

An *imputation* is a payoff vector that is both individually rational and efficient (and thus also feasible). Reasonable to focus on imputations.

The Core

Which imputations incentivise players to form the grand coalition?

Probably the most important solution concept for coalitional games, formalising this kind of *stability* notion, is the so-called 'core' ...

An imputation $\boldsymbol{x} = (x_1, \ldots, x_n)$ is *in the core* of the game $\langle N, v \rangle$ if no coalition $C \subseteq N$ can benefit by breaking away from the grand coalition:

$$\sum_{i \in C} x_i \quad \geqslant \quad v(C)$$

Remark: Individual rationality is a special case of this (with $C = \{i^\star\}$).

D.B. Gillies. Some Theorems on n-Person Games. PhD thesis, Department of Mathematics, Princeton University, 1959.

Example: Game with an Empty Core

Consider the following 3-player TU game $\langle N, v \rangle$, with $N = \{1, 2, 3\}$, in which no single player can generate any surplus on her own:

$$\begin{array}{lll} v(\{1\}) = 0 & v(\{1,2\}) = 7 & v(N) = 8 \\ v(\{2\}) = 0 & v(\{1,3\}) = 6 & \\ v(\{3\}) = 0 & v(\{2,3\}) = 5 & \end{array}$$

For an imputation $\boldsymbol{x} = (x_1, x_2, x_3)$ to be in the core, we must have:

- for stability: $x_1 + x_2 \geqslant 7$ and $x_1 + x_3 \geqslant 6$ and $x_2 + x_3 \geqslant 5$
- for efficiency: $x_1 + x_2 + x_3 = 8$

But this clearly is impossible. So the core is empty.

Question: *What games have a nonempty core? Characterisation?*

Remark: The above game happens to be superadditive but not convex.

Characterisation for Simple Games

Recall: $\langle N, v\rangle$ is a *simple game* if $v(C) \in \{0, 1\}$ for all $C \subseteq N$.

Player $i \in N$ is said to be a *veto player* in the simple game $\langle N, v\rangle$, if for all $C \subseteq N$ it is the case that $i \notin C$ implies $v(C) = 0$.

Proposition 1 *A simple game (and thus also a voting game) has a nonempty core iff it has at least one veto player.*

($\Leftarrow$) Suppose there are $k \geqslant 1$ veto players. Choose imputation $\boldsymbol{x}$ s.t. $x_i = \frac{1}{k}$ if i is a veto player, and $x_i = 0$ otherwise. Then $\boldsymbol{x}$ is in the core:

- for every *winning* coalition C: $\sum_{i \in C} x_i = k \cdot \frac{1}{k} = 1 = v(C)$
- for every *losing* coalition C: $\sum_{i \in C} x_i \geqslant 0 = v(C)$ holds vacuously

($\Rightarrow$) Suppose the core is nonempty and $\boldsymbol{x}$ is in it. Let $i^\star \in \operatorname{argmax}_{i \in N} x_i$. Will show that $i^\star$ can *veto*. By efficiency, $\sum_{i \in N} x_i = 1$. Thus, $x_{i^\star} > 0$. Take any $C \subseteq N$ s.t. $i^\star \notin C$. Then $\sum_{i \in C} x_i < 1$. So $v(C) = 0$. ✓

Convexity is a Sufficient Condition

Recall: $\langle N, v\rangle$ is *convex* if $v(C \cup C') \geqslant v(C) + v(C') - v(C \cap C')$.

Theorem 2 (Shapley, 1971) *Every TU game that is convex has got a nonempty core.*

Proof: Let $[i] := \{1, \ldots, i\}$ for any $i \in N$. Define an imputation $\boldsymbol{x}$:

$$x_i \quad = \quad v([i]) - v([i{-}1])$$

By monotonicity, $x_i \geqslant 0$. And: $\sum x_i = v(N)$, i.e., $\boldsymbol{x}$ is efficient.
Now take any $C \subset N$. Pick $i^\star \in N$ s.t. $[i^\star{-}1] \subseteq C$ but $i^\star \notin C$.
By convexity: $v(C \cup \{i^\star\}) \;\geqslant\; v(C) + v([i^\star]) - v([i^\star{-}1])$. Thus:

$$-v(C) \quad \geqslant \quad x_{i^\star} - v(C \cup \{i^\star\})$$

$$\textstyle\sum_{i\in C} x_i - v(C) \quad \geqslant \quad \sum_{i \in C\cup\{i^\star\}} x_i - v(C \cup \{i^\star\})$$

Repeat: $\sum_{i\in C} x_i - v(C) \;\geqslant\; \cdots \;\geqslant\; \sum_{i\in N} x_i - v(N) \;=\; 0$. ✓

L.S. Shapley. Cores of Convex Games. *Internat. J. Game Theory*, 1:11–26, 1971.

Cohesiveness is a Necessary Condition

Recall: $\langle N, v \rangle$ is *cohesive* if $v(N) \geqslant v(C_1) + \cdots + v(C_K)$ for every possible partition $C_1 \uplus \cdots \uplus C_K = N$.

Proposition 3 *Every TU game with a nonempty core is cohesive.*

Proof: Consider any game $\langle N, v \rangle$ that is *not* cohesive. So there exists a partition $C_1 \uplus \cdots \uplus C_K = N$ with $v(C_1) + \cdots + v(C_K) > v(N)$.

For the sake of contradiction, suppose the core *is* nonempty. So there's an imputation $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $\sum_{i \in C_k} x_i \geqslant v(C_k)$ for all $k \leqslant K$.

Putting everything together, we get:

$$\sum_{i \in N} x_i \;=\; \sum_{i \in C_1} x_i + \cdots + \sum_{i \in C_K} x_i \;\geqslant\; v(C_1) + \cdots + v(C_K) \;>\; v(N)$$

But this contradicts feasibility of $\boldsymbol{x}$, i.e., it cannot be an imputation. ✓

The Bondareva-Shapley Theorem

Thus: convex $\Rightarrow$ has nonempty core $\Rightarrow$ cohesive. *Getting close.*

We can do better and give a *complete characterisation* (w/o proof):

Theorem 4 (Bondareva, 1962; Shapley 1967) *A TU game has a nonempty core iff that game is balanced.*

A collection of weights $\lambda_C \in [0, 1]$, one for each coalition $C \subseteq N$, is called *balanced* if, for all players $i \in N$, we have $\sum_{C \ni i} \lambda_C = 1$.

A TU game $\langle N, v\rangle$ is called *balanced* if, for all balanced collections of weights λ_C, we have $\sum_{C \subseteq N} \lambda_C \cdot v(C) \leqslant v(N)$.

Interpretation: "grand coalition beats dividing time over coalitions"

O.N. Bondareva. The Theory of the Core of an n-Person Game (in Russian). *Vestnik Leningrad University*, 17(13):141–142, 1962.

L.S. Shapley. On Balanced Sets and Cores. *Naval Research Logistics Quarterly*, 14(4):453–460, 1967.

Summary

This has been an introduction to coalitional games, which include:

- *network flow games:* surplus = capacity of coalition's network
- *voting games:* surplus = coalition's ability to win vote

We've modelled them as *transferable-utility games*, i.e., the members of a coalition can freely distribute the surplus amongst themselves.

We've then asked: when will the grand coalition form and be stable? Arguably, if we can find a *payoff vector* that is *in the core*.

So *nonemptiness of the core* is important. Results:

- for *simple (and voting) games:* possible iff there is a veto player
- for *general TU games:* possible iff the game is balanced

What next? Shifting attention from stability to fairness issues.

Exam Preparation

Expect a series of simple questions testing your *familiarity* with and *understanding* of the *concepts* introduced throughout the course.

So these questions will be quite different from the homework questions.

Everything (slides, textbook, papers, homework, lectures, tutorials) is examinable material, but if you are short of time it is best to focus on the *slides* and the corresponding sections in the *textbook*.

This will be a closed-book exam, but you may bring one piece of paper (A4, double-sided) of *handwritten notes* with you.

Recall that the exam counts for 25% of you final grade for the course (with the proviso that you must pass the exam to pass the course).