Game Theory 2025

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Plan for Today

Today we are going to review solution concepts for coalitional games with transferable utility that encode some notion of *fairness:*

- *Banzhaf value:* payoffs should reflect marginal contributions
- *Shapley value:* more sophisticated variant of the same idea
- *Nucleolus:* minimise possible complaints by coalitions

The most important of these is the Shapley value and we are going to use it to exemplify the *axiomatic method* in economic theory.

Part of this is also covered in Chapter 8 of the *Essentials*.

K. Leyton-Brown and Y. Shoham. *Essentials of Game Theory: A Concise, Multidisciplinary Introduction*. Morgan & Claypool Publishers, 2008. Chapter 8.

Reminder: TU Games and Payoff Vectors

A *transferable-utility coalitional game* in characteristic-function form (or simply: a *TU game*) is a tuple $\langle N, v \rangle$, where

- $N = \{1, \ldots, n\}$ is a finite set of *players* and
- $v : 2^N \to \mathbb{R}_{\geqslant 0}$, with $v(\emptyset) = 0$, is a *characteristic function*, mapping every possible *coalition* $C \subseteq N$ to its *surplus* $v(C)$.

Suppose the *grand coalition* N forms. Then we require a *payoff vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geqslant 0}$ to fix what payoff each player should get.

More generally, for fixed N, we may look for a *function* $\boldsymbol{x}$ mapping any given game $\langle N, v \rangle$ to a vector of payments $(x_1(N, v), \ldots, x_n(N, v))$. Any such function constitutes a *solution concept*.

Marginal Contributions

Focus on TU games that are *monotonic* (weakest property we've seen).

Player i increases the surplus of coalition $C \subseteq N \setminus \{i\}$ by the amount $v(C \cup \{i\}) - v(C)$ if she joins. This is her *marginal contribution*.

There are (at least) two ways one could define the 'average' marginal contribution player i makes to coalitions in game $\langle N, v \rangle$ ...

The Banzhaf Value

The *Banzhaf value* gives equal importance to all coalitions in $\langle N, v\rangle$:

$$\beta_i(N,v) \quad = \quad \frac{1}{2^{n-1}} \cdot \sum_{C \subseteq N \setminus \{i\}} v(C \cup \{i\}) - v(C)$$

Note that 2^{n-1} is the number of subsets C of $N \setminus \{i\}$ (normalisation).

This is a solution concept: pick payoff vector $(\beta_1(N,v), \ldots, \beta_n(N,v))$.

Remark: Banzhaf (1965) defined this for the case of voting games.

J.F. Banzhaf III. Weighted Voting Doesn't Work: A Mathematical Analysis. *Rutgers Law Review*, 19(2):317–343, 1965.

Example: Computing the Banzhaf Value

Consider the following 3-player TU game $\langle N, v\rangle$, with $N = \{1,2,3\}$, in which no single player can generate any surplus on her own:

$$
\begin{array}{lll}
v(\{1\}) = 0 & v(\{1,2\}) = 7 & v(N) = 10 \\
v(\{2\}) = 0 & v(\{1,3\}) = 6 & \\
v(\{3\}) = 0 & v(\{2,3\}) = 5 &
\end{array}
$$

Write $\Delta_i(C)$ for the marginal contribution $v(C \cup \{i\}) - v(C)$.

$$
\begin{array}{rcl}
\beta_1(N,v) & = & \frac{1}{4} \cdot (\Delta_1(\emptyset) + \Delta_1(\{2\}) + \Delta_1(\{3\}) + \Delta_1(\{2,3\})) \\
 & = & \frac{1}{4} \cdot (0+7+6+5) \;\; = \;\; \frac{18}{4} \\
\beta_2(N,v) & = & \frac{1}{4} \cdot (0+7+5+4) \;\; = \;\; \frac{16}{4} \\
\beta_3(N,v) & = & \frac{1}{4} \cdot (0+6+5+3) \;\; = \;\; \frac{14}{4}
\end{array}
$$

Exercise: *Arguably, that's fair. But do you see the problem?*

The Shapley Value

The *Shapley value* considers all *sequences* in which the grand coalition may assemble and gives equal importance to each such sequence:

$$\begin{aligned}\varphi_i(N,v) &= \frac{1}{n!} \cdot \sum_{\sigma \in \mathrm{Perm}(N)} [v(\{j \mid \sigma_j \leqslant \sigma_i\}) - v(\{j \mid \sigma_j < \sigma_i\})] \\ &= \frac{1}{n!} \cdot \sum_{C \subseteq N \setminus \{i\}} |C|! \cdot |N \setminus (C \cup \{i\})|! \cdot [v(C \cup \{i\}) - v(C)]\end{aligned}$$

Here $|C|$ players join before i and $|N \setminus (C \cup \{i\})|$ join after her.

Again, $(\varphi_1(N,v), \ldots, \varphi_n(N,v))$ can be considered a payoff vector.

Remark: In simple (and voting) games, for every sequence σ, there will be exactly one player with a nonzero marginal contribution (of 1).

L.S. Shapley. A Value for n-Person Games. In: H.W. Kuhn and A.W. Tucker (eds.), *Contributions to the Theory of Games*, 1953.

Example: Computing the Shapley Value

Consider the following 3-player TU game $\langle N, v\rangle$, with $N = \{1, 2, 3\}$, in which no single player can generate any surplus on her own:

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\begin{array}{lll}
v(\{1\}) = 0 & v(\{1,2\}) = 7 & v(N) = 10 \\
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v(\{3\}) = 0 & v(\{2,3\}) = 5 &
\end{array}
$$

Let $\vec{\Delta}_i(\sigma)$ denote the marginal contribution made by player i when she joins at the point indicated during the sequence σ.

$$
\begin{array}{rcl}
\varphi_1(N, v) & = & \frac{1}{6} \cdot (\vec{\Delta}_1(123) + \vec{\Delta}_1(132) + \vec{\Delta}_1(213) + \cdots + \vec{\Delta}_1(321)) \\
 & = & \frac{1}{6} \cdot (0 + 0 + 7 + 5 + 6 + 5) \;\; = \;\; \frac{23}{6} \\
\varphi_2(N, v) & = & \frac{1}{6} \cdot (7 + 4 + 0 + 0 + 4 + 5) \;\; = \;\; \frac{20}{6} \\
\varphi_3(N, v) & = & \frac{1}{6} \cdot (3 + 6 + 3 + 5 + 0 + 0) \;\; = \;\; \frac{17}{6}
\end{array}
$$

Observe that $\frac{23}{6} + \frac{20}{6} + \frac{17}{6} = 10$ (so this payoff vector is *efficient*).

The Axiomatic Method

Both Banzhaf and Shapley look ok. *So which solution concept is fair?*

An approach to settle such questions is the *axiomatic method:*

- Formulate some fundamental normative properties ('axioms').
- Show that your favourite solution concept satisfies those axioms, and preferably also that it is the only solution concept to do so.

We will go through this exercise for the Shapley value ...

Axioms

What is a good solution concept x mapping any given game $\langle N, v \rangle$ to a vector of payments $(x_1(N,v), \ldots, x_n(N,v))$? Desiderata:

- *Efficiency:* we should have $\sum_{i \in N} x_i(N,v) = v(N)$.
- *Symmetry:* if $v(C \cup \{i\}) = v(C \cup \{j\})$ for all $C \subseteq N \setminus \{i,j\}$, then $x_i(N,v) = x_j(N,v)$ (interchangeable players get equal payoffs).
- *Dummy player:* if $i \in N$ is a 'dummy player' in the sense that $v(C \cup \{i\}) - v(C) = v(\{i\})$ for all coalitions $C \subseteq N \setminus \{i\}$, then we should have $x_i(N,v) = v(\{i\})$.
- *Additivity:* we should have $x_i(N, v_1 + v_2) = x_i(N, v_1) + x_i(N, v_2)$ for the characteristic function $[v_1 + v_2] : C \mapsto v_1(C) + v_2(C)$.

The normative justifications for the first three axioms are convincing. With the additivity axiom some may disagree.

Exercise: *Show that the Shapley value satisfies all four axioms.*

Technical Interlude

Recall that we defined characteristic functions v to be of the form $v : 2^N \to \mathbb{R}_{\geqslant 0}$, i.e., mapping coalitions to non-negative values.

For purely technical reasons, for our proof of the next result, we drop this restriction and work with characteristic functions $v : 2^N \to \mathbb{R}$.

Characterisation Result

Surprisingly, our four axioms fully determine how to divide the surplus:

Theorem 1 (Shapley, 1953) *The Shapley value is the only way of satisfying efficiency, symmetry, dummy player axiom, and additivity.*

Proof: ($\Rightarrow$) We've seen already that φ satisfies the axioms.

($\Leftarrow$) Need to show axioms uniquely fix *some* function $\boldsymbol{x}$.

For *games of the form* $\langle N, \alpha_S \cdot v_S \rangle$ with $S \in 2^N \setminus \{\emptyset\}$, $\alpha_S \in \mathbb{R}$, and $v_S(C) = \mathbb{1}_{C \supseteq S}$, due to *dummy*, *symmetry*, *efficiency* we must have:

$$x_i(N, \alpha_S \cdot v_S) = \tfrac{\alpha_S}{|S|} \text{ for } i \in S \text{ and } x_i(N, \alpha_S \cdot v_S) = 0 \text{ for } i \notin S$$

For *arbitrary games* $\langle N, v \rangle$ with any $v : 2^N \to \mathbb{R}$, observe that v has a unique representation of this form (which we can build 'bottom-up'):

$$v(C) = \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S \cdot v_S(C) \quad \text{with} \quad \alpha_S = v(S) - \sum_{S' \in 2^S \setminus \{\emptyset, S\}} \alpha_{S'}$$

Uniqueness of $\boldsymbol{x}$ for $\langle N, v \rangle$ now follows from *additivity*. ✓

Shapley Value and Stability

How does the Shapley value relate to our stability concepts?

Recall: An *imputation* is a payoff vector that is efficient and IR.

Proposition 2 *For superadd. games, the Shapley value is an imputation.*

Proof: *Efficiency* follows from our axiomatic characterisation. ✓
By superadditivity, $v(C \cup \{i\}) - v(C) \geqslant v(\{i\})$, i.e., all marginal contributions of i are no less than the surplus she can generate alone. The Shapley value is an average over such marginal contributions, so we must have $\varphi_i(N, v) \geqslant v(\{i\})$ (*individual rationality*). ✓

Recall: The *core* is the set of efficient payoff vectors for which no coalition has an incentive to break away from the grand coalition.

Proposition 3 *For convex games, the Shapley value is in the core.*

We omit the proof. It uses a similar idea as the proof we had given to show that the core of a convex game is always nonempty.

The Nucleolus

A solution concept combining stability and fairness considerations . . .

Given imputation $\boldsymbol{x} = (x_1, \ldots, x_n)$, think of $v(C) - \sum_{i \in C} x_i$ as the strength of C's *complaint*. Note: $\boldsymbol{x} \in$ core $\Leftrightarrow$ no complaints > 0

We now want to *minimise complaints* (as we cannot fully avoid them). Let $\vec{c}(\boldsymbol{x})$ be the 2^n-vector of complaints, ordered from high to low.

The *nucleolus* is defined as the set of imputations $\boldsymbol{x}$ for which $\vec{c}(\boldsymbol{x})$ is *lexicographically minimal*. Thus, you first try to avoid the strongest complaint, then the second strongest, and so forth.

Nice properties of the nucleolus (proofs immediate):

- always *nonempty* (unless the set of imputations is empty)
- subset of the *core* (unless the core is empty)

Also: the nucleolus has *at most one* element (difficult proof omitted).

D. Schmeidler. The Nucleolus of a Characteristic Function Game. *SIAM Journal of Applied Mathematics*, 17(6):1163–1170, 1969.

Computational Considerations

Coalitional games give rise to a number of interesting research challenges of a computational nature:

- *Compact representation:* Assuming that v is simply 'given' is unrealistic. We require a compact form of representation.

  (Related to *preference representation languages*, discussed earlier.)

- *Computing solutions:* Some of the solution concepts discussed (e.g., the nucleolus) require sophisticated algorithm design.

- *Coalition formation:* For cohesive games we can assume that the grand coalition will form, and if it does, this is socially optimal.

  But for general games, computing a partition $C_1 \uplus \cdots \uplus C_K = N$ that maximises $\sum_{k \leqslant K} v(C_k)$ is a nontrivial algorithmic problem.

G. Chalkiadakis, E. Elkind, and M. Wooldridge. *Computational Aspects of Cooperative Game Theory*. Morgan & Claypool Publishers, 2011.

Summary

This conclues our review of transferable-utility games. We've seen:

- solution concepts to reflect stability and fairness considerations: core, nucleolus, Banzhaf value, Shapley value
- questions of existence (nonemptiness of the core)
- questions of relationships between concepts
- axiomatic method: normative characterisation of solutions
- computational considerations

What next? Guest lecture by Guido Schäfer.

Course Review

This has been an introduction to game theory, covering these topics:

- strategic games in normal form and Bayesian games
- strategic games in extensive form and imperfect-information games
- auctions and mechanism design ("inverse game theory")
- coalitional games of transferable utility

In a strategic game a *solution* is a profile of strategies. In a coalitional game it is a coalition structure and a choice of who gets what.

A *solution concept* suggests what solutions will emerge in a game. We have focused on *stability*, less so on efficiency and fairness.

We have hardly spoken about *applications* of game theory in other disciplines, but they are there (that's why the field is so successful) and you should try to discover those most relevant to you ...