Game Theory 2025

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Plan for Today

Today we are going to focus on the special case of *zero-sum games* and discuss two positive results that do not hold for games in general.

- new solution concepts: *maximin* and *minimax solutions*
- *Minimax Theorem:* maximin = minimax = NE for zero-sum games
- *fictitious play:* basic model for learning in games
- *convergence result* for the case of zero-sum games

The first part of this is also covered in Chapter 3 of the *Essentials*.

K. Leyton-Brown and Y. Shoham. *Essentials of Game Theory: A Concise, Multidisciplinary Introduction*. Morgan & Claypool Publishers, 2008. Chapter 3.

Zero-Sum Games

Today we focus on *two-player games* $\langle N, \boldsymbol{A}, \boldsymbol{u}\rangle$ with $N = \{1, 2\}$.

Notation: Given player $i \in \{1, 2\}$, we refer to her opponent as $-i$.

Recall: A *zero-sum game* is a two-player normal-form game $\langle N, \boldsymbol{A}, \boldsymbol{u}\rangle$ for which $u_i(\boldsymbol{a}) + u_{-i}(\boldsymbol{a}) = 0$ for all action profiles $\boldsymbol{a} \in \boldsymbol{A}$.

Examples include (but are not restricted to) games in which you can *win* $(+1)$, *lose* (-1), or *draw* (0), such as *Matching Pennies* (left):

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

	L	R
T	5, -5	-3, 3
B	0, 0	2, -2

Constant-Sum Games

A *constant-sum game* is a two-player normal-form game $\langle N, \boldsymbol{A}, \boldsymbol{u}\rangle$ for which there exists a $c \in \mathbb{R}$ such that $u_i(\boldsymbol{a}) + u_{-i}(\boldsymbol{a}) = c$ for all $\boldsymbol{a} \in \boldsymbol{A}$.

Thus: A zero-sum game is a constant-sum game with constant $c = 0$.

Everything about zero-sum games to be discussed today also applies to constant-sum games, but for simplicity we only talk about the former.

Fun Fact: Football is *not* a constant-sum game, as you get 3 points for a win, 0 for a loss, and 1 for a draw. But prior to 1994, when the "three-points-for-a-win" rule was introduced, World Cup games were constant-sum (with 2, 0, 1 points, for win, loss, draw, respectively).

Maximin Strategies

The definitions on this slide apply to arbitrary normal-form games . . .

Suppose player i wants to maximise her worst-case expected utility (e.g., if all others conspire against her). Then she should play:

$$s_i^\star \in \underset{s_i \in S_i}{\operatorname{argmax}} \min_{\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}} u_i(s_i, \boldsymbol{s}_{-i})$$

Any such $s_i^\star$ is called a *maximin strategy* (typically there is just one).

Solution concept: assume each player will play a maximin strategy.

Call $\max\limits_{s_i} \min\limits_{\boldsymbol{s}_{-i}} u_i(s_i, \boldsymbol{s}_{-i})$ player i's *maximin value* (or *security level*).

Exercise: Maximin and Nash

Consider the following two-player game:

	L	R
T	8, 2	0, 0
B	0, 0	8, 4

What is the maximin solution?

How does this relate to Nash equilibria?

<u>Note:</u> This is neither a zero-sum nor a constant-sum game.

Exercise: Maximin and Nash Again

Now consider this fairly similar game, which *is* zero-sum:

	L	R
T	8, -8	0, 0
B	0, 0	8, -8

What is the maximin solution?

How does this relate to Nash equilibria?

Minimax Strategies

Now focus on two-player games only, with players i and $-i$ ...

Suppose player i wants to minimise $-i$'s best-case expected utility (e.g., to *punish* her). Then i should play:

$$s_i^\star \in \operatorname*{argmin}_{s_i \in S_i} \max_{s_{-i} \in S_{-i}} u_{-i}(s_i, s_{-i})$$

Any such $s_i^\star$ is called a *minimax strategy* (typically there is just one).

Call $\min\limits_{s_i} \max\limits_{s_{-i}} u_{-i}(s_i, s_{-i})$ player $-i$'s *minimax value*.

So, by analogy, player i's minimax value is $\min\limits_{s_{-i}} \max\limits_{s_i} u_i(s_i, s_{-i})$.

Remark: An alternative interpretation of player i's minimax value is what she gets when her opponent has to play *first* and i can *respond*.

Equivalence of Maximin and Minimax Values

Recall: For two-player games, we have seen the following definitions.

- Player i's *maximin value* is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.
- Player i's *minimax value* is $\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$.

Lemma 1 *In a two-player game, maximin and minimax value coincide:*

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

Exercise: *Can you see why?*

Interlude

To see that the lemma is not trivial, observe that it becomes *false* if we quantify over actions rather than strategies:

$$\max_{a_i} \min_{a_{-i}} u_i(a_i, a_{-i}) \stackrel{?}{=} \min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i})$$

Take *Matching Pennies* with players being confined to pure strategies. If you go first (LHS) you get -1 but if you go second (RHS) you get 1.

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Proof of Lemma

Let us now prove the lemma. The claim is, for any two-player game:

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

One direction is straightforward:

($\leqslant$) LHS is what i can achieve when she *has to* move first, while RHS is what i can achieve when she *can* move second. ✓

For the full equation, we sketch the proof for 2x2-games only:

	L	R
T	A	B
B	C	D

Rowena's maximin strategy is to play T with probability p so Colin cannot affect her EU: $Ap+C(1-p) = Bp+D(1-p)$

So *maximin value* is $Ap+C(1-p)$ for this p $\Rightarrow$ $\frac{AD-BC}{A-B-C+D}$

Colin's minimax strategy is to play L with probability q so Rowena cannot affect her EU: $Aq+B(1-q) = Cq+D(1-q)$

So *minimax value* is $Aq+B(1-q)$ for this q $\Rightarrow$ $\frac{AD-BC}{A-B-C+D}$

So we really get the same value! ✓ (Exercise: *Verify this!*)

The Minimax Theorem

Recall: A zero-sum game is a two-player game with $u_i(\boldsymbol{a}) + u_{-i}(\boldsymbol{a}) = 0$.

Theorem 2 (Von Neumann, 1928) *In a zero-sum game, a strategy profile is a NE iff each player's expected utility equals her minimax value.*

Proof: Let v_i be the minimax/maximin value of player i (and $v_{-i} = -v_i$ that of player $-i$).

(1) Suppose $u_i(s_i, s_{-i}) \neq v_i$. Then one player does worse than she could (v_i as *maximin* value). So she can deviate: (s_i, s_{-i}) *is not a NE*. ✓

(2) Suppose $u_i(s_i, s_{-i}) = v_i$. Then you cannot do better even if you were allowed to move second (v_i as *minimax* value). So (s_i, s_{-i}) *is a NE*. ✓

John von Neumann (1903–1957)

J. von Neumann. Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, 100(1):295–320, 1928.

Computing Nash Equilibria in Zero-Sum Games

The Minimax Theorem suggests a way of computing Nash equilibria for zero-sum games that is simpler than the general approach.

The reason why this simplifies matters is that, to compute the maximin (or minimax) value of a player, you only need to consider *her* payoffs.

Learning in Games

Suppose you keep playing the same game against the same opponents. You might try to *learn* their *strategies*.

A good hypothesis might be that the *frequency* with which player i plays action a_i is approximately her *probability* of playing a_i.

Now suppose you always best-respond to those hypothesised strategies. And suppose everyone else does the same. *What will happen?*

We are going to see that for *zero-sum games* this process *converges* to a NE. This yields a method for *computing a NE* for the (non-repeated) game: just *imagine* players engage in such "*fictitious play*".

Empirical Mixed Strategies

Given a *history* of actions $H_i^\ell = a_i^0, a_i^1, \ldots, a_i^{\ell-1}$ played by player i in ℓ prior plays of game $\langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$, fix her *empirical mixed strategy* $s_i^\ell \in S_i$:

$$s_i^\ell(a_i) \quad = \quad \underbrace{\frac{1}{\ell} \cdot \#\{k < \ell \mid a_i^k = a_i\}}_{\text{relative frequency of } a_i \text{ in } H_i^\ell} \quad \text{for all } a_i \in A_i$$

Best Pure Responses

Recall: Strategy $s_i^\star \in S_i$ is a *best response* for player i to the (partial) strategy profile $\boldsymbol{s}_{-i}$ if $u_i(s_i^\star, \boldsymbol{s}_{-i}) \geqslant u_i(s_i', \boldsymbol{s}_{-i})$ for all $s_i' \in S_i$.

Due to expected utilities being convex combinations of plain utilities:

Observation 3 *For any given (partial) strategy profile $\boldsymbol{s}_{-i}$, the set of best responses for player i must include at least one pure strategy.*

So we can restrict attention to *best pure responses* for player i to $\boldsymbol{s}_{-i}$:

$$a_i^\star \in \operatorname*{argmax}_{a_i \in A_i} u_i(a_i, \boldsymbol{s}_{-i})$$

Fictitious Play

Take any action profile $\boldsymbol{a}^0 \in A$ for the normal-form game $\langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$.

Fictitious play of $\langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$, starting in $\boldsymbol{a}^0$, is the following process:

- In round $\ell = 0$, each player $i \in N$ plays action a_i^0.
- In any round $\ell > 0$, each player $i \in N$ plays a *best pure response* to her opponents' *empirical mixed strategies:*

$$a_i^\ell \in \underset{a_i \in A_i}{\operatorname{argmax}}\, u_i(a_i, \boldsymbol{s}_{-i}^\ell), \text{ where}$$

$$s_{i'}^\ell(a_{i'}) = \tfrac{1}{\ell} \cdot \#\{k < \ell \mid a_{i'}^k = a_{i'}\} \text{ for all } i' \in N \text{ and } a_{i'} \in A_{i'}$$

Assume some deterministic way of *breaking ties* between maxima.

This yields a sequence $\boldsymbol{a}^0 \twoheadrightarrow \boldsymbol{a}^1 \twoheadrightarrow \boldsymbol{a}^2 \twoheadrightarrow \ldots$ with a corresponding sequence of empirical-mixed-strategy profiles $\boldsymbol{s}^0 \twoheadrightarrow \boldsymbol{s}^1 \twoheadrightarrow \boldsymbol{s}^2 \twoheadrightarrow \ldots$

Question: Does $\lim_{\ell \to \infty} \boldsymbol{s}^\ell$ exist and is it a meaningful strategy profile?

Example: Matching Pennies

Let's see what happens when we start in the upper lefthand corner HH (and break ties between equally good responses in favour of H):

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Any strategy can be represented by a single probability (of playing H).

$$\begin{aligned}
&\text{HH } (\tfrac{1}{1}, \tfrac{1}{1}) \twoheadrightarrow \text{HT } (\tfrac{2}{2}, \tfrac{1}{2}) \twoheadrightarrow \text{HT } (\tfrac{3}{3}, \tfrac{1}{3}) \twoheadrightarrow \text{TT } (\tfrac{3}{4}, \tfrac{1}{4}) \twoheadrightarrow \text{TT } (\tfrac{3}{5}, \tfrac{1}{5})\\
&\twoheadrightarrow \text{TT } (\tfrac{3}{6}, \tfrac{1}{6}) \twoheadrightarrow \text{TH } (\tfrac{3}{7}, \tfrac{2}{7}) \twoheadrightarrow \text{TH } (\tfrac{3}{8}, \tfrac{3}{8}) \twoheadrightarrow \text{TH } (\tfrac{3}{9}, \tfrac{4}{9})\\
&\twoheadrightarrow \text{TH } (\tfrac{3}{10}, \tfrac{5}{10}) \twoheadrightarrow \text{HH } (\tfrac{4}{11}, \tfrac{6}{11}) \twoheadrightarrow \text{HH } (\tfrac{5}{12}, \tfrac{7}{12}) \twoheadrightarrow \cdots
\end{aligned}$$

<u>Exercise:</u> *Can you guess what this will converge to?*

Convergence Profiles are Nash Equilibria

In general, $\lim_{\ell \to \infty} \boldsymbol{s}^\ell$ does not exist (no guaranteed convergence). But:

Lemma 4 *If fictitious play converges, then to a Nash equilibrium.*

Proof: Suppose $\boldsymbol{s}^\star = \lim_{\ell \to \infty} \boldsymbol{s}^\ell$ exists. To see that $\boldsymbol{s}^\star$ is a NE, note that $s_i^\star$ is the strategy that i *seems* to play when she best-responds to $\boldsymbol{s}^\star{}_{-i}$, which she *believes* to be the profile of strategies of her opponents. ✓

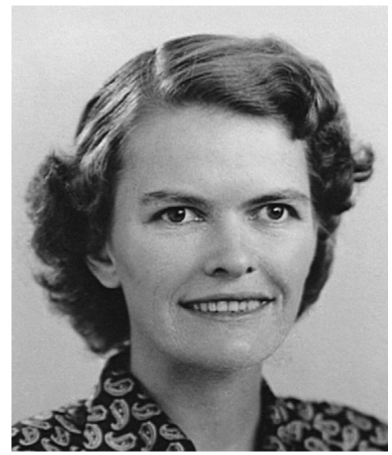
Remark: This lemma is true for arbitrary (not just zero-sum) games.

Convergence for Zero-Sum Games

Good news:

Theorem 5 (Robinson, 1951) *For any zero-sum game and initial action profile, fictitious play will converge to a Nash equilibrium.*

We know that <u>if</u> FP converges, then to a NE.
Thus, we still have to show <u>that</u> it will converge.
The proof of this fact is difficult and we are not going to discuss it here.



Julia Robinson
(1919–1985)

J. Robinson. An Iterative Method of Solving a Game. *Annals of Mathematics*, 54(2):296–301, 1951.

Summary

We have seen that *zero-sum games* are particularly well-behaved:

- *Minimax Theorem:* your expected utility in a Nash equilibrium will simply be your minimax/maximin value
- Convergence of *fictitious play:* if each player keeps responding to their opponent's estimated strategy based on observed frequencies, these estimates will converge to a Nash equilibrium

Both results give rise to alternative methods for computing a NE.

What next? Players who have incomplete information (are uncertain) about certain aspects of the game, such as their opponents' utilities.