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# In Search of Universal Properties of Musical Scales

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## Abstract

Musical scales have both general and culture-specific properties. While most common scales use octave equivalence and discrete pitch relationships, there seem to be no other universal properties. This paper presents an additional property across the world's musical scales that may qualify for universality. When the intervals of 998 (just intonation) scales from the Scala Archive are represented on an Euler lattice, 96.7% of them form star-convex structures. For the subset of traditional scales this percentage is even 100%. We present an attempted explanation for the star-convexity feature, suggesting that the mathematical search for universal musical properties has not yet reached its limits.

## 1. Introduction

What makes a set of notes a musical scale? If one looks up the definition of a scale in *Grove Music Online*, one finds 'A sequence of notes in ascending or descending order of pitch', and also 'a scale is a sequence long enough to define unambiguously a mode, tonality, or some special linear construction'. It is difficult from this and other definitions of a scale to determine what then *exactly* a scale is, and what it is not (Lindley & Turner-Smith, 1993). Some properties of scales have been proposed, such as maximal evenness (Clough & Douthett, 1991), Myhill's property (Clough & Myerson, 1985), well-formedness (Carey & Clampitt, 1989), and cardinality equals variety (Clough & Myerson, 1985), most of which have been defined mathematically. Well-known scales like the diatonic scales or the chromatic 12-tone scale are special in the sense that they possess many of these properties (Balzano, 1980; Agmon, 1989). However, the aforementioned authors also show that none of these properties are required for a set of notes to form a scale, and thus they do not present universal properties and neither do they contribute to the definition of a scale.

Apart from these rather mathematical properties, more intuitive features of scales have been proposed as well. The notes of a scale tend to be arranged asymmetrically within the octave, with some pitch steps bigger than others. The asymmetry offers clues about a melody's tonal centre, letting a listener quickly figure out where the tune is in relation to the tonic note (Browne, 1981; Ball, 2008). Furthermore, the use of discrete pitch relationships, as well as the concept of octave equivalence seem, while not universal in early and prehistoric music (Nettl, 1956; Sachs, 1962), rather common to current musical systems (Burns, 1999).

In this paper, we will give an in-depth examination of two previously proposed scale properties (Honingh & Bod, 2005), convexity and star-convexity, across a wide range of music cultures, and we try to relate these properties to more intuitive features of scales. In contrast to the scale properties listed above, the properties of convexity and star-convexity can be validated both mathematically and empirically. In previous work we showed that for a small collection of scales, all scales were star-convex (Honingh & Bod, 2005). However, since that collection represents only a tiny part of all existing scales, the current work tests the convexity-hypothesis on a much larger dataset of 998 scales from all over the world, and presents an attempted explanation of this (star-)convexity property.

It should be emphasized that we do not wish to present a new definition of the concept 'scale'. Yet we do investigate a condition that can possibly be used to refine

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a definition of a scale. Therefore, we look into a large corpus, the Scala Archive (Scala Archive, 2010), which is a database of scales collected from books, articles, websites and other media.

## 2. Scales visualized on the Euler lattice

### 2.1 Just intonation scales

Just intonation (JI) is a tuning system in which the frequencies of notes are related by integer ratios. Similarly, a just intonation scale is a scale in which every element is defined by an integer ratio. This ratio represents the frequency ratio of the scale tone, with respect to the tonic of the scale. An example of a just intonation scale is the well-known diatonic major scale, which may be represented as 1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, 2/1, in which the element 1/1 represents the tonic. A frequency ratio from a just intonation scale can be expressed as integer powers of primes:

$$2^p \cdot 3^q \cdot 5^r \ldots, \qquad \text{with } p, q, r \in \mathbb{Z}. \tag{1}$$

If the highest prime that is used in a particular just intonation scale is $n$, that scale is called an $n$-limit just intonation scale. An $n$-limit just intonation scale can be visualized on an $n$-dimensional lattice (representing the $n$ -limit just intonation system) as follows. Each axis of the lattice represents a prime and the grid points on the axis represent an integer power. The frequency ratio 5/3 can be written as $2^0 \cdot 3^{-1} \cdot 5^1$ and thus be visualized as point $(0, -1, 1)$ in the three-dimensional lattice representing the first three primes. Since most scales repeat themselves every octave, usually the axis representing the prime 2 can be omitted. In this way five-limit scales can be visualized on a two-dimensional lattice and seven-limit scales on a three-dimensional lattice. Beyond three dimensions, visualization is not possible any more. However, in theory, the higher limit scales can be represented in higher dimensional lattices.

We refer to these lattices as generalized Euler lattices.[1] Usually the Euler lattice is known as the two-dimensional lattice in which one axis is represented by integer powers of the fifth 3/2, and the other axis by integer powers of the major third 5/4 (see Figure 1). This lattice is obtained by applying a basis transformation to the two-dimensional lattice having primes 3 and 5 representing the axes (Honingh & Bod, 2005).

[1]This lattice representation and minor variants of it appear in numerous discussions on tuning systems, for example Von Helmholtz (1863/1954), Riemann (1914), Fokker (1949), and Longuet-Higgins (1962a, 1962b). Fokker (1949) attributes this lattice representation originally to Leonhard Euler, whence Euler lattice.

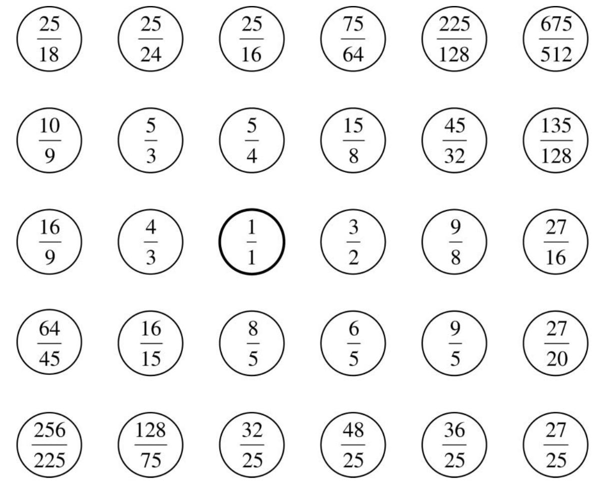
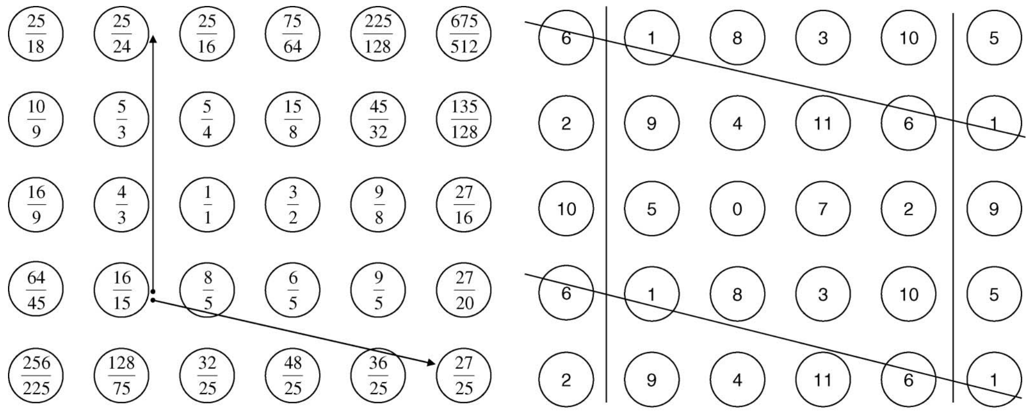



Fig. 1. The two-dimensional Euler lattice where one axis is represented by integer powers of the fifth 3/2, and the other axis by integer powers of the major third 5/4. Only a small part of the two-dimensional Euler lattice is shown. In theory it can be extended infinitely in both the horizontal and the vertical direction. The origin of the lattice is shown in bold.

### 2.2 Equal tempered scales

Not all scales are just intonation scales. In the history of music, the notion of the *equal tempered* scale has existed since the middle ages. Equal temperament is a tuning system in which the octave (or more rarely, another interval) is equally divided into a certain number of intervals. An equal tempered scale is a scale that can be expressed in terms of elements of an equal temperament. Equal tempered scales are usually denoted in terms of cents, where one cent is defined as a hundredth part of a 12-tone equal tempered semitone. In this way, an octave measures 1200 cents. The 12-tone chromatic equal tempered scale can thus be written as: 0, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, 1100. Equal tempered scales cannot be visualized on the Euler lattice, unless they are approximations of a just intonation scale. Fokker's periodicity blocks form a method by which an equal tempered approximation of a just intonation scale can be made (Fokker, 1969). In an Euler lattice as shown in Figure 1 or 2, unison vectors can be found which represent very small ratios, which are known as commas. If ratios that are separated by a comma are treated as being equivalent, the Euler lattice can fold, and the dimension of the lattice reduces by one. Each comma that is identified in this way reduces the dimension of the lattice, such that $n$ commas reduces an $n$-dimensional lattice to zero, which means that the number of pitches in the lattice is finite. This 0-dimensional lattice is called a periodicity block. The $m$ pitches of the periodicity block can be identified with $m$-tone equal temperament and thus gives approximations of the just intonation ratios on




(a) Unison vectors (b) Periodicity block

Fig. 2. (a) Construction of periodicity block from unison vectors. (b) The 12-tone equal temperament, denoted by pitch classes 0 to 11, is created.

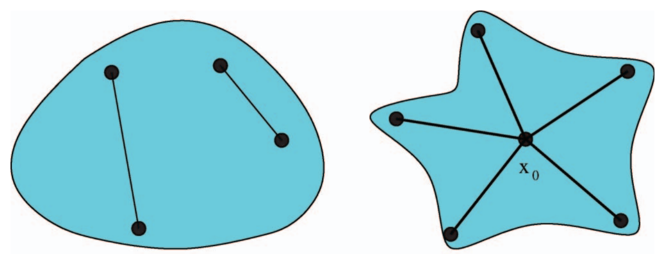
the (original) Euler lattice. In Figure 2, it is shown how a periodicity block is created.

## 3. Convexity and star-convexity

When just intonation scales are represented in the Euler lattice, a very high percentage of them share a surprising property: they form convex shapes. Intuitively this means that these shapes have no coves or holes. An exact definition of convexity in discrete space is not trivial, however. It is convenient to first define convexity for (Euclidean) continuous space: an object is convex if for every pair of points within the object every point of the straight line segment that joins them is also within the object. Star-convexity is related to convexity. An object is star-convex if there exists a point such that the line segment from this point to any other point in the object is contained within the object. If a set is convex, it is also star-convex. See Figure 3 for a two-dimensional representation of these concepts.

In a discrete (Euclidean) space, the concept of (star-) convexity is different from that concept in a continuous space, but the underlying intuition remains the same. In a discrete (two-dimensional Euclidean) space convexity is defined as follows: a set is convex if the convex-hull of the set contains no more than all points of the set. A convex hull of a set $X$ is the minimal convex set – following the definition of a convex set for continuous space – containing $X$. A discrete set is star-convex if there exists a point $x_0$ in the set such that all points (of the set) lying on the line segment from $x_0$ to any point in the set are contained in the set.

We have shown elsewhere that under a basis transformation of the space, a convex set remains convex (Honingh, 2006).




(a) convex (b) star-convex

Fig. 3. A convex and a star-convex object in two dimensional Euclidean space.

## 4. Convex just intonation scales

While we previously demonstrated that the major and minor diatonic scales, and a small selection of other five-limit JI scales, are (star-)convex (Honingh & Bod, 2005), the property was never systematically investigated for all known JI scales. We will use the Scala Archive for this purpose. This archive is one of the largest databases of musical scales in existence and is available as freeware (Scala Archive, 2010). The majority of the scales in Scala are theoretically constructed by music theorists and composers, such as Ellis, Von Oettingen, Fokker, Partch, Wilson, Johnston and Erlich. Other scales correspond to *traditional* scales by which we mean ancient or cultural scales that have been passed down from generation to generation. Scala contains Greek, Arabic, Chinese, Japanese, Korean, Persian, Indian, Vietnamese, Indonesian and Turkish scales.

Two problems need to be considered here. The first is that the traditional scales are mainly based on oral tradition. For a scale to be included in the Scala Archive, the scale needs to be notated in either ratios or cents. As ethnomusicological measurements of scales face various difficulties (Ellis, 1965; Schneider, 2001) this can lead to misinterpretations of the scale. The second problem is that, since no complete definition of a scale exists, and since the Scala Archive scales have been collected from various media, any set of notes that is labelled by somebody as a scale, could have been added to the Scala Archive. As a consequence, the corpus may contain sets of notes on which no overall agreement exists whether or not it serves as a scale. However, one could wonder whether there is *one* correct representation of a scale based on oral tradition, and whether agreement on a scale is possible without a complete definition of a scale. We believe the Scala Archive is still worth using since it is interesting to study sets of notes that are labelled as scales by individuals. The fact that there may be no agreement corresponds with the incomplete existing definitions of a scale.

We have extracted all three-limit, five-limit and seven-limit JI scales from the Scala Archive,[2] forming a test set of 1002 scales. Four scales turned out to have duplicates,[3] which were removed from the dataset, resulting in a total of 998 scales. We wrote Java routines to evaluate all scales on convexity and star-convexity. The test set consisted of two groups, the first containing all three-limit and five-limit scales, and the second group containing the seven-limit scales. The routines calculating the (star-)convexity of the first part included the standard java routine InPolyh.java. The routines calculating the (star-)convexity of the seven-limit scales included routines from the freely available QuickHull3D (Lloyd, 2004). The cases for which seven-limit scales were lying in a plane (instead of in three-dimensional space) were calculated and evaluated separately.

We have tested how many of the 998 three-limit, five-limit and seven-limit just intonation scales of the Scala Archive are convex and star-convex. The results are displayed in Table 1. In total, 86.9% of the tested scales are convex and 96.7% are star-convex. This percentage is high because there is no evidence that composers and music theorists construct their scales deliberately on the basis of star-convexity. While the boundary between theoretically constructed and traditional scales is sometimes hard to establish, it was easy to determine that the 33 non-star-convex scales (out of 998) were clearly theoretically constructed, meaning that 100% of the traditional scales are star-convex. Although the high percentages of convexity and star-convexity contribute to our knowledge about the features of scales, we do not claim that (star-)convexity serves as a necessary property for a scale. Furthermore, it is most definitely not a sufficient condition: not all (star-)convex sets form good scales. Figure 4 presents examples of both traditional and theoretically constructed convex and star-convex scales, visualized in the Euler lattice.

Table 1. The percentage (and number) of the three-, five- and seven-limit just intonation (JI) scales from the Scala Archive that are convex and star-convex.

| | convex | star-convex | total |
|---|---|---|---|
| 3-limit JI scales | 70.0% (21) | 70.0% (21) | 30 |
| 5-limit JI scales | 84.0% (316) | 97.9% (368) | 376 |
| 7-limit JI scales | 89.5% (530) | 97.3% (576) | 592 |
| total | 86.9% (867) | 96.7% (965) | 998 |

The three-limit scales can be visualized on a one-dimensional lattice, the five-limit scales on a two-dimensional lattice and seven-limit scales on a three-dimensional lattice. Somewhat surprisingly, the percentage of (star)-convex scales increases with the dimensionality of the scales: relatively more seven-limit scales are (star-)convex than five-limit scales, and relatively more five-limit scales are (star-)convex than three-limit scales.

In Figure 5, the length distribution of the test set scales is shown. The length of the scales varies from 3 to 171 elements with a mean of 16.8. Most scales are located in the area between 7 and 25 elements. Not every scale has a unique JI representation. For example, for the well-known major scale, both three-limit and five-limit representations exist. In our test set we included all JI representations for each scale as they can be found in Scala.

It may be interesting to see where the convex scales appear in the length distribution of the tested scales. Figure 6(a) shows the number of convex scales per specific length of a scale. Figure 6(b) gives the same result as a percentage of all tested scales. For each specific scale length the percentage of convex scales are given.

One may expect that the scales that are non-(star-) convex are longer than average since the chance that a randomly chosen set is (star-)convex decreases to zero with the number of elements of that set (Honingh & Bod, 2005). However, for most scales with more than 50 elements and for all scales with more than 100 elements, the percentage of convex scales is 100% (Figure 6(b)). Furthermore, the non-star-convex scales are relatively short: the average number of elements being 13.9. The average number of elements of the non-convex scales is 17.4.

## 5. Other convex scales

In Section 2.2 we saw that equal tempered scales, when constructed as Fokker blocks, could be represented in the

[2]The dataset that we used is available from http://staff.science.uva.nl/~ahoningh/data.html

[3]The duplicate pairs are: (1) sal-farabi_diat2.scl and ptolemy_diat.scl, (2) hexany6.scl and smithgw_pel2.scl, (3) hirajoshi2.scl and pelog_jc.scl, (4) ptolemy.scl and zarlino.scl.

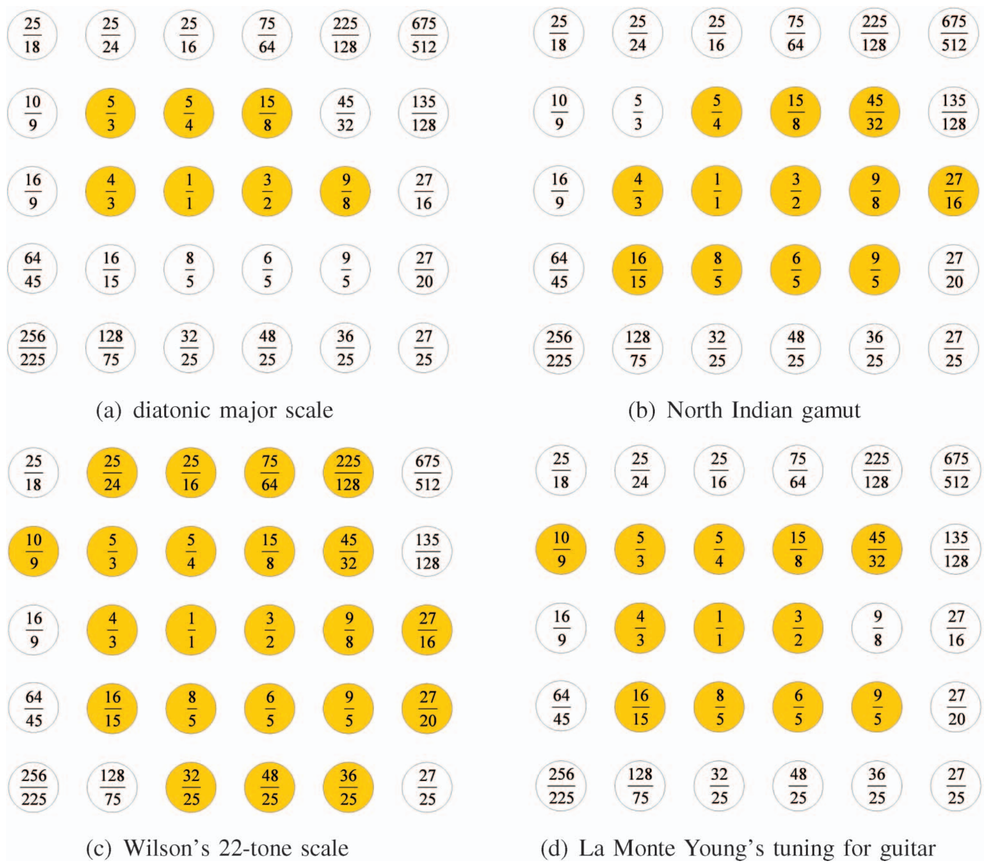



Fig. 4. Four scales on the generalized Euler lattice in two dimensions. Scales (a) and (b) are traditional scales, scales (c) and (d) are invented scales. Scales (a), (b) and (c) are convex, (d) is star-convex.

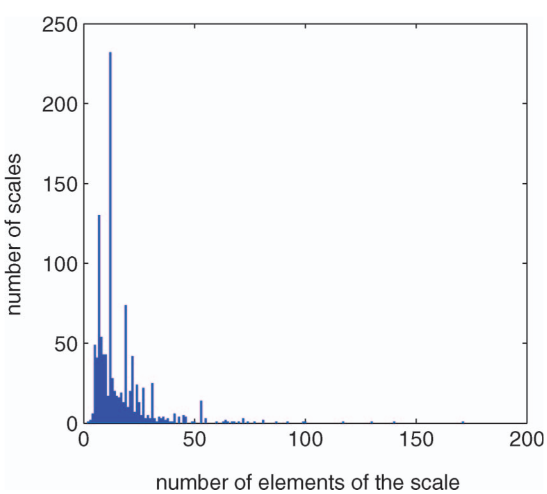



Fig. 5. Length distribution of the tested scales.

Euler lattice. All equal tempered scales that are constructed in this way are necessarily convex since the periodicity block forms the whole space. Note that scales can only be constructed in this way if they are equal to the tuning system in which they are embedded, like for example the familiar 12-tone chromatic scale. Since the Euler lattice is reduced to zero dimensions, actually a new definition of convexity would be needed (Honingh, 2006). However, since the scale is made up of all the elements of the new lattice, it would be impossible for the scale to be non-convex. And if a Fokker block is viewed as a subset of the original Euler lattice, it is clear that Fokker blocks are convex.[4] Still, because it is not possible to evaluate the (star-)convexity of equal tempered scales that cannot be represented as Fokker blocks, equal tempered scales have not been part of the experiments in this paper.

Most of the properties that have been defined for scales, do not apply directly to just intonation scales. Therefore, one cannot compare the property of convexity to other scale properties like well-formedness (Carey & Clampitt, 1989) (which is defined in terms of step-size, and requires the scale to be generated by a single interval), maximal evenness (Clough & Douthett, 1991) (which is defined in terms of pitch classes), and Myhill's

[4]Remember, however, that an equal tempered scale can only be visualized as a Fokker block if it forms an approximation of a just intonation scale. Thus, not all equal tempered scales can be evaluated in terms of convexity.

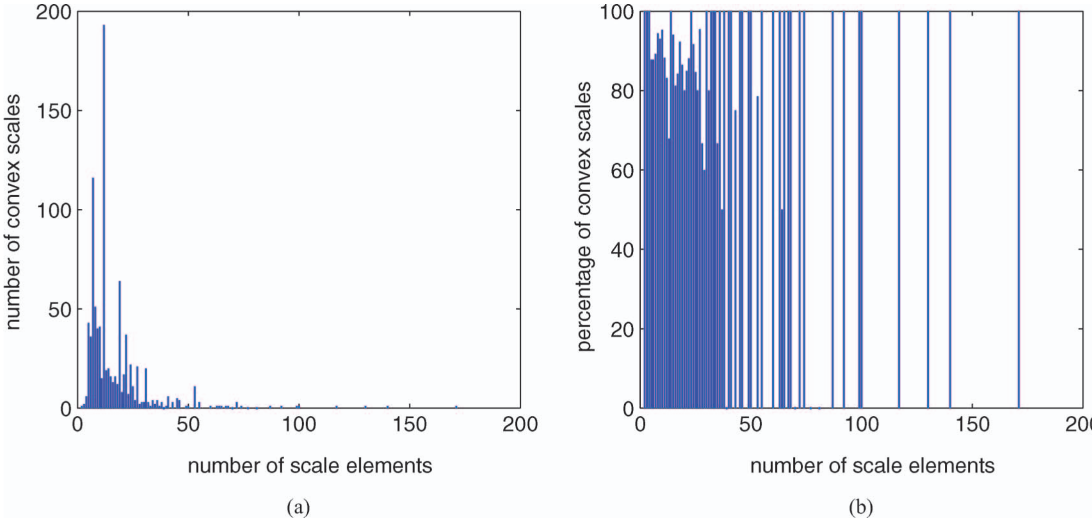



Fig. 6. Number of convex scales (a) and percentage of convex scales (b) per scale length.

property (Clough & Myerson, 1985) (which applies to rank-two regular temperaments).

## 6. Interpretation of (star-)convexity

Obtaining a (star-)convex set by randomly choosing points on a lattice is highly unlikely. It decreases to less than 1% if a (random) scale is longer than three notes in the case of convexity and longer than 10 notes in the case of star-convexity (Honingh & Bod, 2005). Hence we wonder whether we can explain the (star-)convexity of these scales from more general scale properties. Intuition may tell us that it is quite natural for the elements of a scale to be in close connection to one another on the Euler lattice, and thus have a higher chance of forming a convex set. But why is this so?

Sensory consonance seems to be an important factor in scales (Sethares, 1999). Sensory consonance is, at least in Western scales, related to simple integer ratios, and there is an apparent relation between the simplicity of a ratio and the number of elements it passes in the Euler lattice when a straight line is drawn between this ratio and the origin: the simpler the ratio, the fewer elements it passes (however, this statement cannot be turned around). We can therefore imagine that if we choose a number of intervals, starting with the most consonant ones[5] and choosing the most consonant interval that is left every time we choose, we will end up with a star-convex scale which is probably convex as well. Of course, this way of creating scales produces a very limited number of scales without much variety. Yet still, from this example we can infer that the concept of consonance is linked to the concept of convexity, at least for Western scales.

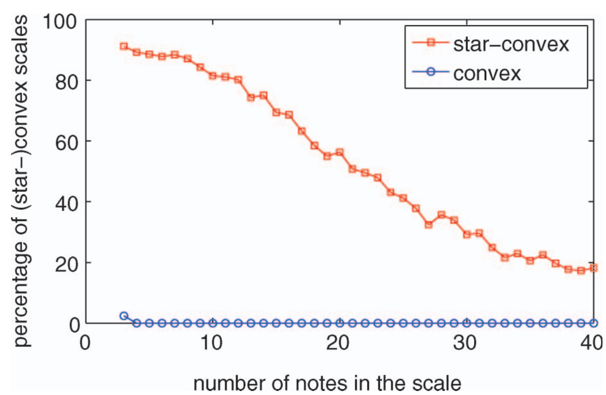



Fig. 7. Using rule 1 (see text for details), the percentages of convex and star-convex scales are shown. For each number of notes, a scale is generated randomly a thousand times.

There is another important property of scales, which is related to the distribution of scale elements. The elements of a scale tend to be arranged not symmetrically within the octave, some pitch steps being larger than others (Ball, 2008). However, this asymmetry does not mean that such arrangement corresponds to a random distribution. Usually, the notes of a scale divide the octave *somewhat* equally: there can be different sizes of scale steps, but not too many. (This is formalized in the Maximal Evenness property (Clough & Douthett, 1991) which applies to scales defined in terms of pitch classes.) It is remarkable that the *Grove Music Online* (Drabkin, 2011), the *Oxford Companion to Music* (Scholes, Nagley, & Temperley, 2011), and the *Oxford Dictionary of Music* ('Scale', 2011) all fail to make notice of the properties of consonance and element distribution when defining a scale.

[5]There is of course not *one* way to do this, since consonance is not unambiguously defined.

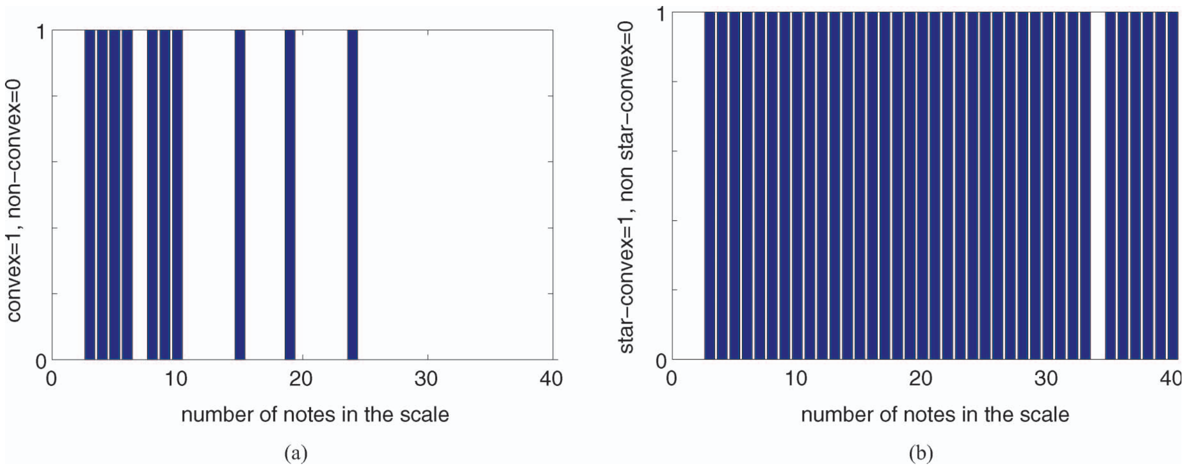



Fig. 8. Using rule 2 (see text for details), the (a) convexity and (b) star-convexity are plotted as a binary property against the scale length.

The property of 'somewhat equal element distribution' is not directly related to convexity, but it restricts the possibilities of forming scales. To investigate whether the properties of distribution and consonance influence the property of convexity, we generated random five-limit just intonation scale elements, from an $m \times m$ Euler lattice, using one of the following two constraints/rules.

1. The octave is divided into $n$ equal intervals, and from each interval (a determined width from the octave, in which a number of ratios from the $m \times m$ Euler lattice fit), a five-limit just intonation ratio is randomly chosen.
2. Having obtained the octave division of $n$ equal intervals, from each interval the most consonant ratio is chosen, according to Euler's Gradus function[6] (which calculates the simplicity of an interval) (Euler, 1739/1865).

Note that the division of the octave into $n$ equal intervals is unrelated to any tuning system or (equal) temperament. It has only been introduced as a method to create a somewhat equal division of five-limit just intonation ratios, representing a scale. Rule 1 takes into account this 'somewhat equal element distribution' property of scales, while rule 2 takes into account the consonance property. Making certain choices, the behaviour of both rules can be shown graphically. That is, we have used a 15 × 15 Euler lattice, and have chosen the number of notes in the scale to run from 3 to 40. In Figure 7 the percentages of convex and star-convex scales are shown when rule 1 is used, and when, for each octave division, a scale is chosen randomly a thousand times. The figure shows that the percentage of convex scales is zero for almost every octave division. The percentage of star-convex scales diminishes when the number of notes in the scale increases. If rule 2 is used, there is no random component any more, and for each number of notes, one scale is returned (according to the rule). Figures 8(a) and (b) show the convexity and star-convexity for scales with 3 to 40 elements. Thus the number of convex scales diminishes when the number of elements increases. For star-convexity, however, there is a high correlation: almost all scales created by rule 2 are star-convex.

It is difficult to draw hard conclusions from these results. We cannot directly address the correlation between the 'somewhat equal element distribution' property and (star-)convexity, and between the consonance property and (star-)convexity, since the two rules above are not a literal translation of these properties. However, we can note the following.

- Rule 1 does not have any implications for the property of convexity of scales.
- For scales with a large number of elements, there is a low chance of randomly obtaining a star-convex scale, even if rule 1 is taken into account.
- For scales with a large number of elements, there is a low chance of obtaining a convex scale, even if rule 2 is taken into account.
- Virtually all scales that are created using rule 2, are star-convex.

[6]Any positive integer $a$ can be written as a unique product $a = p_1^{e_1} \cdot p_2^{e_2} \ldots p_n^{e_n}$ of positive integer powers $e_i$ of primes $p_1 < p_2 < \ldots < p_n$. Euler's Gradus function is defined as:

$$\Gamma(a) = 1 + \sum_{k=1}^{n} e_k(p_k - 1)$$

and for the ratio $x/y$ (which should be given in lowest terms) the value is $\Gamma(x \cdot y)$.

These results show that star-convexity is related to consonance (as defined by Euler). But since rule 2 only generates a limited number of scales, it can at best explain the star-convexity for a small part of our test set.

A further relation between consonance and star-convexity is shown by the following. The centre $x_0$ of a star-convex set is in contact with all elements of the scale, i.e. all elements can be reached with a straight line, such that all elements on the line are within the set. A star-convex set does not necessarily have only one point that can act as a centre. We found that for 90.5% of the five-limit star-convex scales, the defined tonic of the scale is among the notes that can act as $x_0$. For the seven-limit star-convex scales, this turned out to be the case for 98.3%. For the three-limit star-convex scales, any point could serve as $x_0$ since a three-limit (star-) convex scale is visualized as a connected straight line on the Euler lattice. So in most cases, the tonic of the scale can embody the centre of the star-convex set. This means that *the tonic of the scale is among the notes that form most consonant intervals with the other notes of the scale*.

## 7. Conclusions

While several authors have defined and noticed specific properties of certain scales, the property of convexity seems to be especially noteworthy since so many scales possess this property. On average, 86.9% of the tested scales are convex and 96.7% of the scales are star-convex. Of the set of traditional scales, even 100% turned out to be star-convex. Although most just intonation scales are (star-)convex, it turned out to be far from trivial to explain this property, nor can we relate the property to other more well-known features of scales. We have seen that even artificially constructed scales, which circumscribe the major part of the used database, turn out to be star-convex in most of the cases. This is the more surprising since no evidence exists that composers develop their scales following this property.

Finally it may be noteworthy that star-convexity is not unique for musical scales, but seems to be a prevalent property in many other areas of human perception, from language (Gärdenfors & Williams, 2001) to vision (Jaeger, 2009). In this light, the star-convexity of scales may perhaps only be an instantiation of a more general cognitive property for the domain of music.

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## References

Agmon, E. (1989). A mathematical model of the diatonic system. *Journal of Music Theory*, *33*(1), 1–25.

Ball, P. (2008). Facing the music. *Nature*, *453*, 160–162.

Balzano, G.J. (1980). A group theoretical description of 12-fold and microtonal pitch systems. *Computer Music Journal*, *4*(4), 66–84.

Browne, R. (1981). Tonal implications of the diatonic set. *Theory Only*, *5*, 3–21.

Burns, E. (1999). Intervals, scales and tuning. In D. Deutsch (Ed.), *Psychology of Music* (2nd ed.) (pp. 215–264). New York: Academic Press.

Carey, N., & Clampitt, D. (1989). Aspects of well-formed scales. *Music Theory Spectrum*, *11*(2), 187–206.

Clough, J., & Douthett, J. (1991). Maximally even sets. *Journal of Music Theory*, *35*, 93–173.

Clough, J., & Myerson, G. (1985). Variety and multiplicity in diatonic systems. *Journal of Music Theory*, *29*(2), 249–270.

Drabkin, William. (2011). Scale. *Grove Music Online*. *Oxford Music Online*. Retrieved 28 January 2011 from http://www.oxfordmusiconline.com/subscriber/article/grove/music/24691

Ellis, C.J. (1965). Pre-instrumental scales. *Ethnomusicology*, *9*(2), 126–137.

Euler, L. (1739/1865). Tentamen novae theoriae musicae. In *Musique mathématique* (pp. i–vii, 1–215). Paris.

Fokker, A.D. (1949). *Just Intonation*. The Hague: Martinus Nijhoff.

Fokker, A.D. (1969). Unison vectors and periodicity blocks in the three-dimensional (3-5-7) harmonic lattice of notes. In *Proceedings of Koninklijke Nederlandsche Akademie van Wetenschappen*, Amsterdam, the Netherlands, pp. 153–168.

Gärdenfors, P., & Williams, M.A. (2001). Reasoning about categories in conceptual spaces. In *Proceedings of the Fourteenth International Joint Conference of Artificial Intelligence*, Seattle, WA, USA, pp. 385–392.

Honingh, A.K. (2006). *The origin and well-formedness of tonal pitch structures* (PhD thesis). University of Amsterdam, the Netherlands.

Honingh, A.K., & Bod, R. (2005). Convexity and the well-formedness of musical objects. *Journal of New Music Research*, *34*(3), 293–303.

Jaeger, G. (2009). Natural color categories are convex sets. Key note lecture at the Amsterdam Colloquium 2009. Retrieved from http://www2.sfs.uni-tuebingen.de/jaeger/publications/convPca.pdf

Lindley, M., & Turner-Smith, R. (1993). *Mathematical Models of Musical Scales: A New Approach*. Bonn: Verlag für systematische Musikwissenschaft.

Lloyd, J.E. (2004). Quickhull3d. Retrieved from http://people.cs.ubc.ca/~lloyd/java/doc/quickhull3d/quickhull3d/QuickHull3D.html

Longuet-Higgins, H.C. (1962a). Letter to a musical friend. *Music Review*, *23*, 244–248.

Longuet-Higgins, H.C. (1962b). Second letter to a musical friend. *Music Review*, *23*, 271–280.

Nettl, B. (1956). Scale and melody. In *Music in Primitive Culture* (pp. 45–60). Cambridge: Harvard University Press.

Riemann, H. (1914). Ideen zu einer Lehre von den Tonvorstellungen. *Jahrbuch der Musikbibliothek Peters*, *21/22*, 1–26.

Sachs, C. (1962). *The Wellsprings of Music*. The Hague: Martinus Nijhoff.

Scala Archive. (2010). Retrieved July 2010 from http://www.huygens-fokker.org/docs/scales.zip (version 74).

Scale. (2011). In M. Kennedy (Ed.), *The Oxford Dictionary of Music* (2nd ed. rev.). Retrieved 28 January 2011 from http://www.oxfordmusiconline.com/subscriber/article/opr/t237/e8999

Schneider, A. (2001). Sound, pitch, and scale: From 'tone measurements' to sonological analysis in ethnomusicology. *Ethnomusicology*, *45*(3), 489–519.

Scholes, P., Nagley, J., & Temperley, N. (2011). Scale. In A. Latham (Ed.), *The Oxford Companion to Music*. Retrieved 28 January 2011 from http://www.oxfordmusiconline.com/subscriber/article/opr/t114/e5921

Sethares, W.A. (1999). *Tuning, Timbre, Spectrum, Scale* (2nd ed.). Berlin: Springer.

Von Helmholtz, H. (1863/1954). *On the Sensations of Tone* (2nd English ed.). New York: Dover.