The Herbrand Topos

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Category theory, categorical logic, topos theory
Constructivism
Proof theory
What is a constructive proof?

Theorem (Euclid)

There are infinitely many prime numbers.

Proof.

First of all, 2 is prime. Now suppose $p_1, \ldots, p_n$ is a list of prime numbers. Consider $u = p_1 \ldots p_n + 1$ and let $v$ be the smallest divisor of $u$ bigger than 1. Then $v$ is prime and different from all $p_i$.

Theorem

Either $e + \pi$ or $e - \pi$ is irrational.

Proof.

Suppose both $e + \pi$ and $e - \pi$ are irrational. Then also $(e + \pi) + (e - \pi) = 2e$ is rational. Contradiction!
What is a constructive proof?

**Theorem (Skolem-Mahler-Lech)**

The indices of the null elements of a linear recurrence sequence are the union of a finite set and finitely many arithmetic progressions.

**Theorem**

There is a *non-principal ultrafilter* on the natural numbers, a collection $\mathcal{U}$ of sets of natural numbers such that:

(i) $\mathcal{U}$ does not contain any finite set;

(ii) for any subset $A$ of $\mathbb{N}$, either $A$ or its complement belongs to $\mathcal{U}$;

(iii) if two sets belong to $\mathcal{U}$, then so does their intersection.

But note: the axiom of choice is needed to prove the second result, and even with this axiom we cannot write down a formula in the language of set theory defining a specific non-principal ultrafilter.
In 1908 he wrote a paper “Over de onbetrouwbaarheid der logische principes”. He concluded that traditional logic carries some of the blame for the non-constructive nature of classical mathematics. He objected in particular to the Law of Excluded Middle ($\varphi \lor \neg \varphi$).
Brouwer goes further

Brouwer actually argued for principles contradicting usual mathematics:

**Theorem?**

All functions $f : \mathbb{R} \to \mathbb{R}$ are continuous. And so are functions $\Phi : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ (for the Baire and discrete topology, respectively).

Acceptance of such principles makes you an *intuitionist*.

Other constructivists have argued for:

**Theorem?**

All functions $f : \mathbb{N} \to \mathbb{N}$ are computable. And if $(\forall x \in \mathbb{N}) (\exists y \in \mathbb{N}) \varphi(x, y)$, then there is a computable $f : \mathbb{N} \to \mathbb{N}$ such that $(\forall x \in \mathbb{N}) \varphi(x, f(x))$ (“Church’s Thesis”).

Acceptance of such principles makes you a *Russian-style constructivist*. 
Topos theory

Does this still make sense?

We need a notion of an alternative mathematical universe in which only the laws of intuitionistic logic hold. A beautiful such notion is provided by category theory and *topos theory* in particular (see next semester’s lecture by Jaap van Oosten).

Topos theory has its origins in algebraic geometry and the theory of sheaves (Grothendieck). Surprisingly, there is a sheaf topos in which Brouwer’s theorem saying that all functions $f: \mathbb{R} \to \mathbb{R}$ are continuous, is valid.
Effective topos

But then there is also a topos in which all $f: \mathbb{N} \to \mathbb{N}$ are computable and in which Church’s Thesis holds: the effective topos due to Martin Hyland.

The effective topos is used in computer science to provide models of programming languages (for example, the Calculus of Constructions).

In the effective topos there are no nonstandard models of arithmetic, because:

**Theorem (McCarty)**

Church’s Thesis implies that there are no nonstandard models of arithmetic.
What’s this talk about?

There is topos closely related to the effective topos in which there are nonstandard models of arithmetic: the Herbrand topos. In this topos Church’s Thesis fails (of course), but the following principle holds:

**Bounded Church’s thesis**

If $(\forall x \in \mathbb{N}) (\exists y \in \mathbb{N}) \varphi(x, y)$, then there is a computable $f: \mathbb{N} \to \mathbb{N}$ such that $(\forall x \in \mathbb{N}) (\exists y \leq f(x)) \varphi(x, y)$.

Unfortunately, I will not be able to show you the topos. But I will give you a subcategory of the Herbrand topos, the *Herbrand assemblies*, which will still give you the consistency of Bounded Church’s Thesis with the existence of nonstandard models.
Notation

For $A \subseteq \mathbb{N}$, put

$$P_{\text{fin}}(A) = \{ n \in \mathbb{N} : n \text{ codes a finite set all whose elements belong to } A \},$$

and

$$T(A) = \{ \alpha \in P(P_{\text{fin}}(A)) : \alpha \text{ is non-empty and upwards closed } \},$$

where upwards closed means

$$m \in \alpha, m \subseteq n \implies n \in \alpha.$$
Herbrand assemblies

A Herbrand assembly is a triple \((X, A, \alpha)\) where \(X\) is a set, \(A\) is a subset of \(\mathbb{N}\) and \(\alpha\) is a function \(X \rightarrow TA\). A morphism of Herbrand assemblies \(f: (X, A, \alpha) \rightarrow (Y, B, \beta)\) is a function \(f: X \rightarrow Y\) for which there is a computable function \(\varphi\) such that:

1. the function \(\varphi\) is defined on every \(n \in P_{\text{fin}}(A)\) and its result belongs to \(P_{\text{fin}}(B)\).
2. if \(x \in X\) and \(n \in \alpha(x)\), then \(\varphi(n) \in \beta(f(x))\).

But why is this a bit like the category of sets? And how could one interpret logic in this category?
Ubuntu philosophy in the words of Bruce Bartlett

A thing is a thing only in the way that it relates to other things. No man is an island. You exist only through, and you are completely determined by, your connections with others. You are nothing more than the sum of your relationships.

Can you formulate usual set-theoretic constructions in such relational terms?
Some categorical notions

- terminal object, initial object
- product, sum, exponential
- subobjects
- pullback
- natural numbers object

All these notions exist both in the category of sets and in the Herbrand assemblies.
Both in the category of sets and in the Herbrand assemblies, we have:

- if $X$ is an object, then $\text{Sub}(X)$ is a Heyting algebra.
- if $f: Y \to X$, then the pullback functor $f^*: \text{Sub}(X) \to \text{Sub}(Y)$ preserves this structure.
- This pullback functor has both adjoints satisfying the Beck-Chevalley condition.

But what does this have to do with logic?
Categorical analysis of logic

When I say *logic*, I mean multi-sorted logic with empty sorts and in which the sorts are closed under products. Then I have:

1. a collections of sorts.
2. a collection of terms, going from one sort to another.
3. for every sort a collection of predicate of that sort closed under the logical operations like disjunction and conjunction.
4. for every term $f: Y \to X$ a substitution operation sending predicates on $X$ to predicates on $Y$.
5. both adjoints (Lawvere: quantifiers as adjoints), satisfying Beck-Chevalley.

So now it’s obvious, right?
The nonstandard model

There are two Herbrand assemblies: $N = (\mathbb{N}, \mathbb{N}, \gamma)$ and $M = (\mathbb{N}, \mathbb{N}, \delta)$, with

\[
\gamma(n) = \{ m : n \in m \}, \quad \delta(n) = P_{\text{fin}}(\mathbb{N}).
\]

These are not isomorphic.

$N$ is the natural numbers object in the Herbrand assemblies, while $M$ is a nonstandard model.

What are the maps $f : N \rightarrow N$ in the Herbrand assemblies?
Some suggestions for further reading


On constructivism:


Some suggestions for further reading

On category theory:


On categorical logic:


Some suggestions for further reading

On topos theory:


