Conceptions of set

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Concepts

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- Roughly speaking, a criterion of application for a concept `C` tells us to which objects `C` applies.

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- No need to take a stand on what concepts are, but typically agreed they are associated with a *criterion of application*.
- Roughly speaking, a criterion of application for a concept C tells us to which objects C applies.
- Certain concepts are thought to be also associated with a *criterion of identity*.
- A criterion of identity specifies the conditions under which some thing x falling under a concept C is the same as another thing y, also falling under C.

Ranges of application and disapplication

- Call the *range of application* of a concept the class of objects to which a given concept applies.
- And call the *range of disapplication* of a concept the class of things to which the concept *disapplies*—where 'disapplies' is used as an antonym of 'applies'.

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- And call the *range of disapplication* of a concept the class of things to which the concept *disapplies*—where 'disapplies' is used as an antonym of 'applies'.
- A criterion of application and a criterion of identity for a concept need not settle all questions concerning, respectively, whether the concept applies or disapplies to a certain thing and the identity between objects falling under that concept.

Smidget

- Soames (1999) considers the concept [x: x is a smidget], associated with the following criterion of application:

(1) 'smidget' applies to x if x is greater than four feet tall

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- Clearly, $[x\colon x \text{ is a smidget}]$ is such that its range of application and its range of disapplication do not exhaust all possibilities: if x is three feet tall 'smidget' neither applies nor disapplies to it.

Waismann

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  Suppose I have to verify a statement such as 'There is a cat next door'; suppose I go over to the next room, open the door, look into it and actually see a cat. Is this enough to prove my statement? . . . What . . . should I say when the creature later on grew to a gigantic size? Or if it showed some queer behaviour usually not to be found with cats, say, if, under certain conditions it could be revived from death whereas normal cats could not? Shall I, in such a case, say that a new species has come into being? Or that it was a cat with extraordinary properties? (Waismann 1945: 121–122)

A cat?

Open-texture

- Waismann offers the following diagnosis of the situation:

> *The fact that in many cases there is no such thing as a conclusive verification is connected to the fact that most of our empirical concepts are not delimited in all possible directions. (Waismann 1945: 122)*

- Most of our empirical concepts, Waismann says, display *open-texture*.

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  > *The fact that in many cases there is no such thing as a conclusive verification is connected to the fact that most of our empirical concepts are not delimited in all possible directions. (Waismann 1945: 122)*

- Most of our empirical concepts, Waismann says, display *open-texture*.
- Translated into the terminology we are adopting to say that a concept displays open-texture seems to amount to saying that the range of application and the range of disapplication of a concept do not exhaust all possibilities.

Shapiro on open-texture

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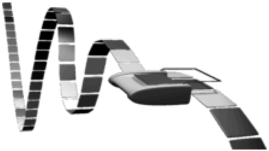
> *in the thirties, and probably for some time afterward, [the pre-theoretic notion of computability] was subject to open-texture. The concept was not delineated with enough precision to decide every possible consideration concerning tools and limitations. (Shapiro 2006: 441)*



Shapiro on open-texture

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- We can shed further light on Shapiro's suggestion by looking at it in terms of the distinction between *concepts* and *conceptions*.

Concepts and conceptions

Mary

Mary is going to be rewarded by her company for her successful work on a certain case. Jane and Susan, however, disagree over whether this reward is fair: Jane thinks that Mary's work has, in fact, entirely been carried out by Mary's colleague Marianne, whilst Susan is persuaded that it has not.

Concepts and conceptions

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Jill

Like Mary, Jill is going to be rewarded by her company for her work on a certain case. Jane and Susan, however, disagree over whether this decision is fair: according to Jane, Mary should be rewarded because it is fair to reward employees depending on their contribution, whereas according to Susan it is not. For Susan, a company should reward its employees depending on their efforts in their work, regardless of its outcome.

Concepts and conceptions

- In the Jill example, the disagreement concerns the criterion of application for ‘fair’.
- But now recall that it seems reasonable to assume that both Jane and Susan *possess* the concept of fairness: we can say that although both Jane and Susan have the concept of fairness, they have different *conceptions* of it.

Concepts and conceptions

- In the Jill example, the disagreement concerns the criterion of application for 'fair'.
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Conception

A *conception* of `C`, where `C` is a concept, is a (possibly partial) answer to the question 'What is it to be something falling under `C`?' which someone could disagree with without being reasonably deemed not to possess `C`.

- N.B.: the distinction need not be *sharp* and need not be *fixed*.

Back to Shapiro

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- Shapiro's suggestion accords with our account of the concept/conception distinction.
- The concept of computability—the *pre-theoretic notion*, as Shapiro puts it—displayed open-texture.
- In order to put the concept to mathematical use, the concept needed to be sharpened by putting forward a *conception* of computability, and various candidates presented themselves: computability as effective computability, but also, for instance, computability as *practicable* computability.
- Eventually, the mathematical community settled for computability as effective computability, on the basis of, *inter alia* considerations about the interest and fruitfulness of this notion.

The naïve conception

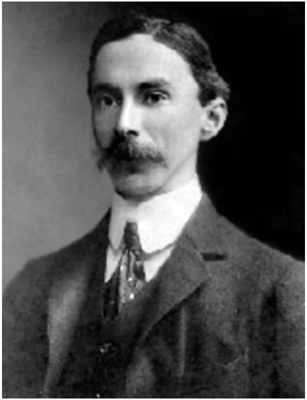
- According to the *naïve conception of set*, sets are extensions of predicates.
- This is usually taken as sanctioning the *Naïve Comprehension Schema*, stating that every condition expressible in the language of our theory determines a set:

(NC) $$\exists y \forall x (x \in y \leftrightarrow \Phi(x)),$$

where $\Phi(x)$ is any formula in $\mathcal{L}_\in$ in which x is free and which contains no free occurrences of y.

The naïve conception

- We all know (NC) to be (classically) inconsistent.
- But what about restricting (NC) according to consistency maxims?
- The material that follows is based on Incurvati & Murzi forthcoming.



Quine

- According to Quine, (NC) embodies the only really intuitive conception of set, and should, in keeping with the maxim of minimal mutilation, be restricted as little as possible so as to avoid the paradoxes.

Only because of Russell's paradox and the like do we not adhere to the naïve and unrestricted comprehension schema [...] Having to cut back because of the paradoxes, we are well advised to mutilate no more than what may be fairly responsible for the paradoxes' (Quine 1951: 50–51)

Maddy's Zermelo

- According to Maddy (1988), Zermelo (1908) saw himself as giving us as much of (NC) as possible without inconsistency.

- With Zermelo, Maddy suggests, it emerged the *one step back from disaster* rule of thumb: if a natural principle leads to contradiction, the principle should be weakened just enough to block the contradiction.

Goldstein

- A similar suggestion has recently been made by Laurence Goldstein:

The naive comprehension axiom captures a very clear conception of sets, and it makes sense to tamper with it as little as possible, identifying and justifying the restriction that must be made to its application, rather than building up set theory from entirely new foundations. (Goldstein 2013: 35-6; see also Goldstein 2006)

Consistency

- If classical logic is to be preserved, Goldstein's idea amounts to restricting (NC) according to the following injunction:

Consistent

An instance ϕ of (NC) is admissible just in case $\phi \nvdash \perp$.

- But: easy to exhibit instances of (NC) which are consistent given extremely weak assumptions—and therefore admissible according to *Consistent*—but together lead to inconsistency.

Example 1

Example 1

The following are instances of (NC):

(1) $\exists y \forall x (x \in y \leftrightarrow x = x)$;

(2) $\forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge x \notin x))$.

- (1) has a model consisting of just one object a such that $a = \{a\}$.(2) is true in any model consisting of a single hereditarily finite set. Hence, (1) and (2) are both admissible according to *Consistent*.
- But it is routine to derive a contradiction from (1) and (2) together.

A simple lemma

- Possible suggestion: ban instances of (NC) that are obtained using parameters.
- Second example shows that this is not enough to dispense with the problem.
- We first need:

Lemma 1.

For each ϕ in $\mathcal{L}_{\in}$,

(i) $\vdash \exists y \forall x (x \in y \leftrightarrow (\neg\phi \wedge x \notin x)) \rightarrow \phi$;

(ii) $\exists y \forall x (x \notin y) \vdash \phi \rightarrow \exists y \forall x (x \in y \leftrightarrow (\neg\phi \wedge x \notin x))$.

Example 2

Example 2

Take your favourite σ such that both σ and $\neg\sigma$ are consistent—for instance, let σ be shorthand for 'There are at least thirty objects'. Then, the following are instances of (NC):

(3) $\exists y \forall x (x \in y \leftrightarrow (\neg\sigma \wedge x \notin x))$;

(4) $\exists y \forall x (x \in y \leftrightarrow (\sigma \wedge x \notin x))$.

- By Lemma 1(ii), any proof of a contradiction from (3) could be transformed, in the presence of the Empty Set Axiom, into a proof of a contradiction from σ, and similarly for (4). Hence, in the presence of the Empty Set Axiom, (3) and (4) are both admissible instances of (NC) according to *Consistent*.
- But by Lemma 1(i), (3) entails σ, and (4) entails $\neg\sigma$. Contradiction.

Maximal consistency

- The naïve set theorist might take the naïve principles to be so intuitive that we should keep as many instances of them as we consistently can.
- More formally, the acceptable instances of (NC) must form a *maximally consistent* set:

Maximally Consistent

A set of instances Γ of (NC) is admissible just in case $\Gamma \nvdash \perp$ and if Λ is a set of instances of (NC) and properly extends Γ, then $\Lambda \vdash \perp$.

Maximal consistency

- The approach was once advocated by Horwich for truth:

> *[I]t suffices for us to concede that certain instances of [the T-Schema] are not to be included as axioms of the minimal theory, and to note that the principles governing our selection of excluded instances are, in order of priority: (a) that the minimal theory not engender 'liar-type' contradictions; (b) that the set of excluded instances be as small as possible; and ... (c) that there be a constructive specification of the excluded instances that is as simple as possible. (Horwich 1998: 42)*

Maximal consistency

- McGee (1992):

1. Maximally consistent sets of instances of the T-Schema are not recursively axiomatizable (given Q), and hence it is unclear whether they can be said to be *theories* of truth.
2. There exist several maximally consistent, and yet incompatible, sets of instances of the T-Schema.

Generalizing McGee's result

- If T is a theory, we write '$\Gamma \vdash_{\mathsf{T}} \phi$' for '$\Gamma \cup \mathsf{T} \vdash \phi$'. And we make use of the following definitions:

Definition 2.

Let Γ be a set of sentences and Σ a first-order schema.

(i) Γ is T-*consistent* iff $\Gamma \nvdash_{\mathsf{T}} \bot$.

(ii) Γ is Σ-*maximally* T-*consistent* iff Γ is T-consistent and any set which properly extends Γ and contains Σ-instances not in Γ is T-inconsistent.

(iii) Γ is *maximally* T-*consistent* iff it is T-consistent and any set which properly extends Γ is T-inconsistent.

Generalizing McGee's result

- Let S be a consistent first-order theory entailing the axioms of some theory S′, Σ be a first-order schema, and $\mathcal{L}$ be a countable first-order language. Relatively interpret in $\mathcal{L}$ the language obtained by adding to the language of S any non-logical expression in Σ.

Theorem 3.

Suppose that for each $\phi \in \mathcal{L}$ there is an instance Σ_ϕ of Σ such that

$$\vdash_{\mathsf{S}'} \phi \leftrightarrow \Sigma_\phi.$$

Then:

(1) *For any S-consistent set Δ of sentences in $\mathcal{L}$, there is a Σ-maximally S-consistent set Γ of Σ-instances such that $\Gamma \vdash_{\mathsf{S}} \delta$ for every $\delta \in \Delta$.*

(2) *If Ξ is a Σ-maximally S-consistent set of Σ-instances, then, for every $\psi \in \mathcal{L}$, $\Xi \vdash_{\mathsf{S}'} \psi$ or $\Xi \vdash_{\mathsf{S}'} \neg\psi$.*

Applying the result to our case

- Let R be some consistent theory entailing the axioms of the theory whose sole non-logical axiom is

  (NC$_\emptyset$) $$\exists y \forall x (x \in y \leftrightarrow x \neq x).$$

  And suppose that we have relatively interpreted the language of R in a countable first-order language $\mathcal{L}'$. We have:

Lemma 4.

For each $\phi \in \mathcal{L}'$,

$$\vdash_{\mathsf{NC}_\emptyset} \phi \leftrightarrow \exists y \forall x (x \in y \leftrightarrow (\neg \phi \wedge x \notin x)).$$

Applying the result to our case

- Together with Theorem 3, Lemma 4 immediately yields:

Corollary 5.

(1) *For any* R*-consistent set* Δ *of sentences in* $\mathcal{L}'$*, there is a* (NC)*-maximally* R*-consistent set* Γ *of* (NC)*-instances such that* $\Gamma \vdash_{\mathsf{R}} \delta$ *for every* $\delta \in \Delta$*.*

(2) *If* Ξ *is a* (NC)*-maximally* R*-consistent set of* (NC)*-instances, then, for every* $\psi \in \mathcal{L}'$, $\Xi \vdash_{\mathsf{NC}_\emptyset} \psi$ *or* $\Xi \vdash_{\mathsf{NC}_\emptyset} \neg\psi$*.*

First upshot

- Take any theory which interprets Q and contains a set of instances of (NC) which includes $NC_\emptyset$ and is (NC)-maximally consistent with Q.
- The second part of Corollary 5 tells us that any such theory is negation complete, and hence cannot be recursively axiomatized.
- Clearly, however, any set theory worth its name should interpret Q. Moreover, $NC_\emptyset$, which simply asserts the existence of the empty set, should be an acceptable instance of (NC) if anything is.
- Thus, there is no reasonably strong recursively axiomatizable theory that satisfies the injunction dictated by *Maximally Consistent*: to identify our theory of sets with all instances of (NC) that are admissible according to *Maximally Consistent* would seem to fall short of providing a set *theory*.

Second upshot

- The first part of Corollary 5 gives us a recipe for constructing several admissible, and yet incompatible, sets of instances of (NC), each of which has an equal claim to embody the naïve but consistent conception of set on offer. In particular:

Recipe

Take a σ such that σ and $\neg\sigma$ are (individually) consistent with the existence of the empty set. Then, start with the sets $\{\exists y \forall x(x \in y \leftrightarrow (\neg\sigma \wedge x \notin x)), \mathrm{NC}_{\emptyset}\}$ and $\{\exists y \forall x(x \in y \leftrightarrow (\sigma \wedge x \notin x)), \mathrm{NC}_{\emptyset}\}$, and use Corollary 5 to extend each of these sets to a (NC)-maximally consistent set of (NC)-instances. The resulting two (NC)-maximally consistent sets are incompatible given Lemma 4.

Second upshot

- In order to restore consistency, the naïve but consistent set theorist must settle on one such (NC)-maximally consistent set, as embodying her conception.
- Yet, we are given no guidance at all as to how this choice is to be made: consistency was meant to be our sole guide for restricting (NC).
- The provision of several mutually incompatible tamperings of the naïve conception of set seems to amount to the provision of no conception at all.

Some references

- Incurvati 2012, 'How to be a minimalist about sets', *Philosophical Studies* 159: 69–87.
- Incurvati 2014, 'The graph conception of set', *The Journal of Philosophical Logic* 43: 181–208.
- Incurvati & Murzi forthcoming, 'Maximally consistent sets of instances of Naïve Comprehension', *Mind*. Preprint available on my publications page.

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