The Multiverse and its Logics

Benedikt Löwe

Logic, Language and Computation.
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Georg Cantor (1845–1918)
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**Set Theory** was developed as a mathematical theory of sets that later developed into a foundational theory for all of mathematics. As usual with mathematical theories, there was an expectation that natural set-theoretic problems are solvable ("für uns gibt es kein Ignorabimus und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht").
David Hilbert (1862–1943)

The first problem was: is the set of all real numbers uncountable? Or, in other words: does every uncountable set of real numbers have the cardinality of the set of all real numbers?

Theorem (Cohen). If $M \models \text{ZFC}$, then there are $N$ and $N'$ such that $M \subseteq N$ and $M \subseteq N'$ and $N \models \text{ZFC} + \text{CH}$ and $N' \models \text{ZFC} + \neg \text{CH}$.
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Problems for the 20th century posed at the *International Congress of Mathematicians* in Paris, 1900. The first problem was:

$$2^{\aleph_0} = \aleph_1?$$

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\[ N \models \text{ZFC} + \text{CH} \text{ and } N' \models \text{ZFC} + \neg \text{CH}. \]
The multiverse view vs. the universe view

Joel D. Hamkins
The universe view is the commonly held philosophical position that there is a unique absolute background concept of set, instantiated in the corresponding absolute set-theoretic universe, the cumulative universe of all sets, in which every set-theoretic assertion has a definite truth value. On this view, interesting set-theoretic questions, such as the continuum hypothesis and others, have definitive final answers.

The multiverse view [...] holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist. [...] In particular, I shall argue [...] that the question of the continuum hypothesis is settled on the multiverse view by our extensive, detailed knowledge of how it behaves in the multiverse.
The *multiverse view*.
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The *set theoretic multiverse* is the collection of all models of set theory. Between these models, there are relations that tell us how one of them was constructed from another or what models know about each other.
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One example of such a construction method is Cohen’s method of *forcing*:

![Paul Cohen (1934–2007)](image)

**Theorem (Cohen).** If $M \models \text{ZFC}$, then there are $N$ and $N'$ such that $M \subseteq N$ and $M \subseteq N'$ and

$$N \models \text{ZFC} + \text{CH} \text{ and } N' \models \text{ZFC} + \neg \text{CH}.$$
Forcing.

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**Theorem** (Cohen). If \( M \models \text{ZFC} \), then there are \( N \) and \( N' \) such that \( M \subseteq N \) and \( M \subseteq N' \) and

\[
N \models \text{ZFC} + \text{CH} \quad \text{and} \quad N' \models \text{ZFC} + \neg \text{CH}.
\]

In general, forcing is a technique that takes a model of set theory \( V \) and produces a new bigger model \( V[G] \) called a *generic extension*. This construction has the properties that the original model \( V \) is a definable inner model of \( V[G] \) called the *ground model* and that the ground model can express statements about the existence of generic extensions.
If $V, W \models \text{ZFC}$, then we say that $W$ is a *generic extension of $V$* if there is a $\mathbb{P} \in V$ and some $G \in W$ which is $\mathbb{P}$-generic over $V$ such that $W = V[G]$. We say that $V$ is a *ground of $W$*. The *generic multiverse of $V$* consists of the closure of $V$ under the operations of generic extension and ground.
The generic multiverse.

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The generic multiverse can be seen as a directed graph.

Or, slightly more generally:

The *generic multiverse with inner models of $V$* is the closure of $V$ under the operations of generic extension, ground, and inner model. It comes as a graph-structure with two edge relations (interacting with each other).
Provability Logic (1).

If we interpret $\square \varphi$ as “$\varphi$ is provable in PA”, we obtain the provability interpretation:
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\square (\square \varphi \rightarrow \varphi) \rightarrow \square \varphi.
\]  

(Löb)
Provability Logic (1).

If we interpret $\Box \varphi$ as "$\varphi$ is provable in PA", we obtain the provability interpretation:

$$\Box(\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi.$$  \hfill (Löb)

The modal logic $\mathbf{GL}$ is obtained from $\mathbf{K}$ by including all instances of (4) and (Löb).
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The modal logic $\mathbf{GL}$ is obtained from $\mathbf{K}$ by including all instances of (4) and (Löb).

**Theorem** (Segerberg-de Jongh-Kripke, 1971). The set of modal formulas valid in all transitive and conversely well-founded frames is $\mathbf{GL}$. 
A function from the language of modal logic into the set of arithmetical sentences is called a realization if:

- \( R(\bot) = \bot \)
- \( R(\neg \phi) = \neg R(\phi) \)
- \( R(\phi \lor \psi) = R(\phi) \lor R(\psi) \)
- \( R(2\phi) = PA \vdash R(\phi) \).

Theorem (Solovay, 1976). A modal formula is in \( GL \) if and only if all of its realizations are PA-provable.
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\begin{align*}
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\end{align*}
\]
Provability Logic (2).

Robert M. Solovay

A function from the language of modal logic into the set of arithmetical sentences is called a \textit{realization} if

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\end{align*}

\textbf{Theorem} (Solovay, 1976). A modal formula is in \textbf{GL} if and only if all of its realizations are PA-provable.
Hamkins translations and the modal logic of forcing.

A function \( H \) from the language of modal logic into set-theoretic sentences is called a Hamkins translation if:

\[
\begin{align*}
H(\bot) &= \bot \\
H(\neg \phi) &= \neg H(\phi) \\
H(\phi \lor \psi) &= H(\phi) \lor H(\psi) \\
H(2\phi) &= \forall B(J H(\phi) K B = 1 B)
\end{align*}
\]

Question. What is the modal logic of those modal formulas whose Hamkins translations are ZFC-provable?

Definition. The Modal Logic of Forcing \( MLF \) is the set of \( \phi \) such that for all Hamkins translations \( H \), ZFC \( \vdash H(\phi) \).
Hamkins translations and the modal logic of forcing.

A function $H$ from the language of modal logic into set-theoretic sentences is called a **Hamkins translation** if

\[
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H(\bot) &= \bot \\
H(\neg \varphi) &= \neg H(\varphi) \\
H(\varphi \vee \psi) &= H(\varphi) \vee H(\psi) \\
H(\Box \varphi) &= \forall B \left( [H(\varphi)]_B = 1_B \right).
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**Definition.** The **Modal Logic of Forcing** $MLF$ is the set of $\varphi$ such that for all Hamkins translations $H$, $\text{ZFC} \vdash H(\varphi)$. 
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\[ \square \varphi \rightarrow \varphi \]  \hspace{1cm} (T)

\[ \square \varphi \rightarrow \square \square \varphi \]  \hspace{1cm} (4)

\[ \Diamond \square \varphi \rightarrow \square \Diamond \varphi \]  \hspace{1cm} (.2)

\[ \Diamond \square \varphi \rightarrow \varphi \]  \hspace{1cm} (5)

Theorem (Hamkins). The modal logic of forcing MLF contains S4, but not S5.

Theorem (Stavi-Väänänen / Hamkins). There is a model \( M \models ZFC \) in which every instance of (5) holds.


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**Theorem** (Hamkins). The modal logic of forcing $\text{MLF}$ contains $\text{S4.2}$, but not $\text{S5}$.
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**Theorem** (Hamkins). The modal logic of forcing \textbf{MLF} contains \textbf{S4.2}, but not \textbf{S5}.

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What is the modal logic of forcing? (2)
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**Theorem** (Hamkins-Löwe). The modal logic of forcing is exactly $S4.2$.

Generalizations I.

If $\forall \phi = \text{ZFC}$, then we can consider $\text{MLF}_{\forall} = \{ \phi; \text{for all Hamkins translations } H, \forall = H(\phi) \}$. The Stavi-Väänänen/Hamkins result says that there is a model $\forall$ such that $\text{MLF}_{\forall} = S_5$.

Theorem (Hamkins-Löwe). $\text{MLF}_{L} = S_4^2$.

Using the techniques of the main theorem, it is easy to see that $S_4^2 \subseteq \text{MLF}_{\forall} \subseteq S_5$ for any model $\forall$.

Question. Can you find $\forall$ such that $\text{MLF}_{\forall}$ is any modal logic strictly between $S_4^2$ and $S_5$?
Generalizations I.

If $V \models ZFC$, then we can consider $MLF_V := \{ \varphi ; \text{for all Hamkins translations } H, \ V \models H(\varphi) \}$. 
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Using the techniques of the main theorem, it is easy to see that \( S4.2 \subseteq \text{MLF}_V \subseteq S5 \) for any model \( V \).

**Question.** Can you find \( V \) such that \( \text{MLF}_V \) is any modal logic strictly between \( S4.2 \) and \( S5 \)?
Generalizations II: Reversing the arrows (1).

In our introduction, we said that the generic multiverse of a model $V$ was the closure of $V$ under set-generic extensions and ground models. But the Modal Logic of Forcing only talks about set-generic extensions. What if we reverse the direction of our accessibility relation:

A function $G$ from propositional modal logic into the set of sentences of the language of set theory is called a ground translation if

$G(\bot) = \bot$

$G(\neg \phi) = \neg G(\phi)$

$G(\phi \lor \psi) = G(\phi) \lor G(\psi)$

$G(\exists \phi) = G(\phi)$

holds in all grounds.

The modal logic of grounds is the set $\text{MLG} := \{ \phi \mid \text{ZFC} \vdash G(\phi) \}$ for all ground translations $G$. 
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\]

The modal logic of grounds is the set $\text{MLG} := \{ \varphi ; \text{ZFC} \vdash G(\varphi) \}$ for all ground translations $G$. 
Generalizations II: Reversing the arrows (2).

The situation for **MLG** is quite different from that of **MLF**: in **L**, we have that $\Box p \leftrightarrow \Diamond p \leftrightarrow p$. In particular, the modal logic of grounds in **L** is much stronger than **S5**.
The situation for \textbf{MLG} is quite different from that of \textbf{MLF}: in \textbf{L}, we have that $\Box p \leftrightarrow \Diamond p \leftrightarrow p$. In particular, the modal logic of grounds in \textbf{L} is much stronger than $\textbf{S5}$.

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**Theorem.** There are models $V_0$, $V_1$, and $V_2$ such that

$$
\text{MLF}_{V_0} = S4.2 \text{ and } \text{MLG}_{V_0} = S4.2; \\
\text{MLF}_{V_1} = S5 \text{ and } \text{MLG}_{V_1} = S4.2; \text{ and} \\
\text{MLF}_{V_2} = S4.2 \text{ and } \text{MLG}_{V_2} = S5.
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\text{MLF}_{V_2} = S4.2 \text{ and } \text{MLG}_{V_2} = S5.
\]

**Theorem.** It is impossible to have $\text{MLF}_V = S5$ and $\text{MLG}_V = S5$. 
A function \( \mathcal{I} \) from propositional modal logic into the set of sentences of the language of set theory is called an inner model translation if 
\[
\mathcal{I}(\bot) = \bot \quad \mathcal{I}(\neg \varphi) = \neg \mathcal{I}(\varphi) \\
\mathcal{I}(\varphi \lor \psi) = \mathcal{I}(\varphi) \lor \mathcal{I}(\psi) \\
\mathcal{I}(2 \varphi) = \mathcal{I}(\varphi)
\]
holds in all inner models. The modal logic of inner models is the set
\[
\text{MLIM} := \{ \varphi \in \text{ZFC} \mid \mathcal{I}(\varphi) \text{ for all inner model translations } \mathcal{I} \}
\]
As opposed to the conditions "in all generic extensions" and "in all grounds", "in all inner models" is not first-order definable in the language of set theory, so the definition of \( \text{MLIM} \) requires more meta-mathematical care.
A function $I$ from propositional modal logic into the set of sentences of the language of set theory is called a *inner model translation* if

\[
\begin{align*}
I(\bot) &= \bot \\
I(\neg \varphi) &= \neg I(\varphi) \\
I(\varphi \lor \psi) &= I(\varphi) \lor I(\psi) \\
I(\Box \varphi) &= I(\varphi)
\end{align*}
\]

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\[
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\]

\[
I(\square \varphi) = I(\varphi)
\]

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The *modal logic of inner models* is the set

\[
\text{MLIM} := \{ \varphi ; \text{ZFC} \vdash I(\varphi) \text{ for all inner model translations } I \}.
\]
Generalizations III: Other modalities (1).

A function $I$ from propositional modal logic into the set of sentences of the language of set theory is called an *inner model translation* if

$$
I(\bot) = \bot
$$

$$
I(\neg \varphi) = \neg I(\varphi)
$$

$$
I(\varphi \lor \psi) = I(\varphi) \lor I(\psi)
$$

$$
I(\Box \varphi) = I(\varphi)
$$

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The *modal logic of inner models* is the set

$${\text{MLIM}} := \{ \varphi ; \text{ZFC} \vdash I(\varphi) \text{ for all inner model translations } I \}.$$ 

As opposed to the conditions “in all generic extensions” and “in all grounds”, “in all inner models” is not first-order definable in the language of set theory, so the definition of $\text{MLIM}$ requires more meta-mathematical care.
It is easy to see that $S_4.2 \subseteq MLIM$, but since the inner model modality can jump out of the generic multiverse, the phenomenon of "bottomless" models does not replicate.

$\exists \phi \left( \phi \leftrightarrow \phi \right) \land \left( \exists \neg \phi \leftrightarrow \neg \phi \right)$ (Top)

$S_4.2_{Top}$ is the modal logic obtained from $S_4.2$ by adding all instances of (Top).

Theorem (Inamdar-Löwe). $MLIM = S_4.2_{Top}$. 
It is easy to see that $\mathbf{S4}_2 \subseteq \mathbf{MLIM}$, but since the inner model modality can jump out of the generic multiverse, the phenomenon of "bottomless" models does not replicate.

$\mathbf{S4}_2\text{Top}$ is the modal logic obtained from $\mathbf{S4}_2$ by adding all instances of (Top).

Theorem (Inamdar-Löwe). $\mathbf{MLIM} = \mathbf{S4}_2\text{Top}$. 
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\[ \Diamond(((\Box \varphi \leftrightarrow \varphi) \land (\Box \neg \varphi \leftrightarrow \neg \varphi)) \]  

(Top)
It is easy to see that $S4.2 \subseteq MLIM$, but since the inner model modality can jump out of the generic multiverse, the phenomenon of “bottomless” models does not replicate.

$$\Diamond((\square \varphi \leftrightarrow \varphi) \land (\square \neg \varphi \leftrightarrow \neg \varphi)) \quad \text{(Top)}$$

$S4.2\text{Top}$ is the modal logic obtained from $S4.2$ by adding all instances of (Top).
It is easy to see that $\textbf{S4.2} \subseteq \textbf{MLIM}$, but since the inner model modality can jump out of the generic multiverse, the phenomenon of “bottomless” models does not replicate.

$$\diamond((\square \varphi \leftrightarrow \varphi) \land (\square \neg \varphi \leftrightarrow \neg \varphi))$$

$\textbf{S4.2Top}$ is the modal logic obtained from $\textbf{S4.2}$ by adding all instances of (Top).

**Theorem** (Inamdar-Löwe). $\textbf{MLIM} = \textbf{S4.2Top}$. 
A function $S$ from propositional modal logic into the set of sentences of the language of set theory is called a symmetric extension translation if $S(\bot) = \bot$, $S(\neg \phi) = \neg S(\phi)$, $S(\phi \lor \psi) = S(\phi) \lor S(\psi)$, and $S(2\phi) = S(\phi)$ holds in all symmetric extensions.

$\text{MLS} := \{ \phi \mid ZFC \vdash S(\phi) \}$ for all symmetric extension translations $S$.

Theorem (Block). $\text{MLS} = S4$.2.
A function $S$ from propositional modal logic into the set of sentences of the language of set theory is called a **symmetric extension translation** if

$$
S(\bot) = \bot \\
S(\neg \varphi) = \neg S(\varphi) \\
S(\varphi \lor \psi) = S(\varphi) \lor S(\psi) \\
S(\Box \varphi) = S(\varphi)
$$

holds in all symmetric extensions.
Generalizations III: Other modalities (3).

A function $S$ from propositional modal logic into the set of sentences of the language of set theory is called a \textit{symmetric extension translation} if

\begin{align*}
S(\bot) &= \bot \\
S(\neg \varphi) &= \neg S(\varphi) \\
S(\varphi \lor \psi) &= S(\varphi) \lor S(\psi) \\
S(\Box \varphi) &= S(\varphi) \quad \text{holds in all symmetric extensions.}
\end{align*}

\textbf{MLS} := \{ \varphi ; \text{ZFC} \vdash S(\varphi) \text{ for all symmetric extension translations } S \}.
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**Theorem** (Block). **MLS** = S4.2.