The Multiverse and its Logics

Benedikt Löwe

Logic, Language and Computation.
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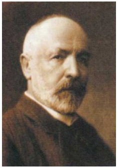
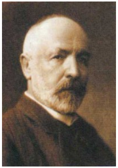


Georg Cantor (1845–1918)



Georg Cantor (1845–1918)

Set Theory was developed as a mathematical theory of sets that later developed into a foundational theory for all of mathematics. As usual with mathematical theories, there was an expectation that natural set-theoretic problems are solvable ("für uns gibt es kein Ignorabimus und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht").

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Theorem (Cohen). If $M \models \mathrm{ZFC}$, then there are N and N' such that $M \subseteq N$ and $M \subseteq N'$ and

$$N \models \mathrm{ZFC} + \mathrm{CH} \text{ and } N' \models \mathrm{ZFC} + \neg\mathrm{CH}.$$

The *multiverse view* vs. the *universe view*

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J. D. Hamkins, "The set-theoretic multiverse," *Review of Symbolic Logic* 5 (2012), pp. 416-449.

The universe view is the commonly held philosophical position that there is a unique absolute background concept of set, instantiated in the corresponding absolute set-theoretic universe, the cumulative universe of all sets, in which every set-theoretic assertion has a definite truth value. On this view, interesting set-theoretic questions, such as the continuum hypothesis and others, have definitive final answers.

The multiverse view [...] holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist. [...] In particular, I shall argue [...] that the question of the continuum hypothesis is settled on the multiverse view by our extensive, detailed knowledge of how it behaves in the multiverse.

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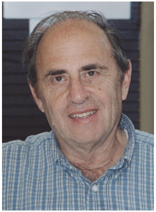
One example of such a construction method is Cohen's method of *forcing*:



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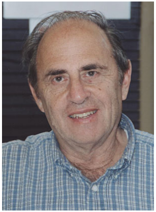
Forcing.



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In general, forcing is a technique that takes a model of set theory V and produces a new bigger model $V[G]$ called a *generic extension*. This construction has the properties that the original model V is a definable inner model of $V[G]$ called the *ground model* and that the ground model can express statements about the existence of generic extensions.

The generic multiverse.

If $V, W \models$ ZFC, then we say that W is a *generic extension of V* if there is a $\mathbb{P} \in V$ and some $G \in W$ which is $\mathbb{P}$-generic over V such that $W = V[G]$. We say that V is a *ground of W*. The *generic multiverse of V* consists of the closure of V under the operations of generic extension and ground.

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The generic multiverse can be seen as a directed graph.

Or, slightly more generally:

The *generic multiverse with inner models of* V is the closure of V under the operations of generic extension, ground, and inner model. It comes as a graph-structure with two edge relations (interacting with each other).

Provability Logic (1).

If we interpret $\Box\varphi$ as "φ is provable in PA", we obtain the provability interpretation:

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Theorem (Segerberg-de Jongh-Kripke, 1971). The set of modal formulas valid in all transitive and conversely well-founded frames is **GL**.

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A function from the language of modal logic into into the set of arithmetical sentences is called a *realization* if

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R(\bot) &= \bot \\
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Theorem (Solovay, 1976). A modal formula is in **GL** if and only if all of its realizations are PA-provable.

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Definition. The *Modal Logic of Forcing* **MLF** is the set of φ such that for all Hamkins translations H, ZFC $\vdash H(\varphi)$.

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J. Stavi, J. Väänänen, “Reflection principles for the continuum”, in: Y. Zhang (ed.), Logic and algebra, Volume 302 of Contemporary Mathematics, American Mathematical Society, 2002, pp. 59-84.

J. D. Hamkins, “A simple maximality principle”, *Journal of Symbolic Logic* 68 (2003), pp. 527-550.

What is the modal logic of forcing? (2)

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Theorem (Hamkins-Löwe). The modal logic of forcing is exactly **S4.2**.

J. D. Hamkins, B. Löwe, "The Modal Logic of Forcing", *Transactions of the American Mathematical Society* 360 (2008), pp. 1793-1817

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Question. Can you find V such that $\mathbf{MLF}_V$ is any modal logic strictly between $\mathbf{S4.2}$ and $\mathbf{S5}$?

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The *modal logic of grounds* is the set **MLG** $:= \{\varphi\,;\, \text{ZFC} \vdash G(\varphi)$ for all ground translations $G\}$.

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The situation for **MLG** is quite different from that of **MLF**: in **L**, we have that $\Box p \leftrightarrow \Diamond p \leftrightarrow p$. In particular, the modal logic of grounds in **L** is much stronger than **S5**.

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J. D. Hamkins, B. Löwe, Moving up and down in the generic multiverse, in: Kamal Lodaya (ed.), Logic and Its Applications, 5th International Conference, ICLA 2013, Chennai, India, January 10-12, 2013, Proceedings Springer-Verlag, Berlin 2013 [Lecture Notes in Computer Science 7750], pp. 139-147

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Theorem. There are models V_0, V_1, and V_2 such that

$$\mathbf{MLF}_{V_0} = \mathbf{S4.2} \text{ and } \mathbf{MLG}_{V_0} = \mathbf{S4.2};$$
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Theorem. It is impossible to have $\mathbf{MLF}_V = \mathbf{S5}$ and $\mathbf{MLG}_V = \mathbf{S5}$.

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A function I from propositional modal logic into the set of sentences of the language of set theory is called a *inner model translation* if

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As opposed to the conditions "in all generic extensions" and "in all grounds", "in all inner models" is not first-order definable in the language of set theory, so the definition of **MLIM** requires more meta-mathematical care.

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T. C. Inamdar, B. Löwe, The Modal Logic of Inner Models, submitted

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Theorem (Inamdar-Löwe). **MLIM** = **S4.2Top**.

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Theorem (Block). $\mathbf{MLS} = \mathbf{S4.2}$.