A Proof-Theoretical View of Collective Rationality

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A discursive dilemma

Consider the following example of discursive dilemma: three individuals i_1, i_2, and i_3 express their opinions about propositions in the agenda $\{a, b, a \wedge b, \neg a, \neg b, \neg(a \wedge b)\}$

	a	$a \wedge b$	b	$\neg a$	$\neg(a \wedge b)$	$\neg b$
i_1	1	1	1	0	0	0
i_2	1	0	0	0	1	1
i_3	0	0	1	1	1	0
maj.	1	0	1	0	1	0

Each agent has a consistent set of propositions. However, by majority, the collective set $\{a, b, \neg(a \wedge b)\}$ is not consistent.

Inferring the contradiction

- The fact that $\{a, b, \neg(a \wedge b)\}$ is not consistent means (syntactically) that we can infer a contradiction from the assumption that a, b and $\neg(a \wedge b)$ hold.
- In this paper, we want to take a closer look at the reasoning steps that are required in order to infer the contradiction.
- We can infer the contradiction by reasoning in classical logic as follows.

$$
\dfrac{\dfrac{\dfrac{}{\{i_1, i_2\} \vdash a}\text{majority} \quad \dfrac{}{\{i_1, i_3\} \vdash b}\text{majority}}{\{i_1, i_2\}, \{i_1, i_3\} \vdash a \wedge b} R\wedge \quad \dfrac{\dfrac{}{\{i_2, i_3\} \vdash \neg(a \wedge b)}\text{majority}}{\{i_2, i_3\}, a \wedge b \vdash \emptyset} \neg L}{\{i_1, i_2\}, \{i_1, i_3\}, \{i_2, i_3\} \vdash \emptyset} \text{cut}
$$

Overview

- Linear logic as an analysis of inferences
- A proof-theoretical definition of individual and collective rationality
- Judgment aggregation: fragments of linear logic
- Judgment aggregation: different logics for individual and collective reasoning
- Possibility and impossibility results for the majority rule
- Extensions and axiomatic treatment
- Conclusion

Linear Logic: resource-sensitive account of reasoning

(Girard, 1987). In sequent calculus for classical logic, *structural rules* of contraction, weakening (and exchange) define how to deal with hypotheses in a proof:

$$\frac{\Gamma, A, A, \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{C} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \mathrm{C}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ \mathrm{W}$$

They determine the behavior of logical connectives, in particular they make the following two presentations of logical rules are equivalent:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ \wedge \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge$$

(*multiplicative and additive presentation*)

- By rejecting structural rules, we are lead to define two conjunctions with different behavior: copying contexts ($\otimes$) or identifying them (&).

Sequent calculus for LL

The language of (multiplicative additive) linear logic (MALL) is defined as follows. Let $\mathcal{A}$ be a set of propositional atoms and $a \in \mathcal{A}$.

$$L ::= a \mid \neg L \mid L \otimes L \mid L \parr L \mid L \,\&\, L \mid L \oplus L$$

A sequent is an expression $\Gamma \vdash \Delta$ where Γ and Δ are multisets of occurrences of formulas in LL.

Identities

$$\frac{}{A \vdash A}\ \text{ax} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{cut}$$

Negation

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ L\neg \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ R\neg$$

Sequent calculus for LL: multiplicative and additives

Multiplicatives

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ \otimes L \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}\ \otimes R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'}\ \parr L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ \parr R$$

Additives

$$\frac{\Gamma, A_i \vdash \Delta}{\Gamma, A_0 \& A_1 \vdash \Delta}\ \& L \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta}\ \& R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\ \oplus L \qquad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_0 \oplus A_1, \Delta}\ \oplus R$$

Discursive dilemmas

- Consider again the discursive dilemma. By dropping W and C, the contradiction may no longer be derivable.
- If the group reasons in LL, the non-logical axioms are again: $\{i_1, i_2\} \vdash a$, $\{i_1, i_3\} \vdash b$ and $\{i_2, i_3\} \vdash \neg(a \,\&\, b)$.

$$\dfrac{\dfrac{\{i_1, i_2\} \vdash a \qquad \{i_1, i_3\} \vdash b}{\{i_1, i_2\}, \{i_1, i_3\} \vdash a \otimes b}\, R\otimes \qquad \{i_2, i_3\} \vdash \neg(a \wedge b)}{\cdots}$$

- The group can infer $a \otimes b$ by using two different coalitions, however $a \,\&\, b$ cannot be inferred by any sequence of winning coalition:

If $\neg(a \wedge b)$ is interpreted as the mutliplicative conjunction, then we have inferred again a contradiction. However, if $\neg(a \wedge b)$ is interpreted additively, then in LL $a \otimes b$ and $\neg(a \,\&\, b)$ are *not* inconsistent, because $a \otimes b, \neg(a \,\&\, b) \nvdash_{LL} \emptyset$.

The Model

- Let N be a (finite) set of agents. An *agenda* $\mathcal{X}_{\mathcal{L}}$ is a (finite) set of propositions in the language $\mathcal{L}_L$ of a given logic L that is closed under complements. i.e. non-double negations.
- We slightly rephrase the usual rationality conditions on judgment sets in terms of sequents derivability.
- A *judgement set* J is a subset of $\mathcal{X}_L$ such that J is (wrt L) *consistent* ($J \nvdash_L \emptyset$), *complete* (for all $\phi \in \mathcal{X}_L$, $\phi \in J$ or $\neg\phi \in J$) and *deductive closed* (if $J \vdash_L \phi$ and $\phi \in \mathcal{X}_L$, then $\phi \in J$).
- Denote $J(\mathcal{X}_L)$ the set of all judgement sets on $\mathcal{X}_L$.
- A *profile* of judgements sets $\mathbf{J}$ is a vector $(J_1, \ldots, J_n)$, where $n = |N|$.

Individual and Collective rationality

- We want to model aggregators that take profiles of judgments sets that are rational wrt L and return a set of judgement which can be evaluated wrt a (possibly) different logic L'.
- In case L and L' are the same, we are in the standard situation in JA. In case the languages of L and L' are different, we need to define a translation function from the language of L into the language of L'.
- A *translation* is a function that maps formulas of one language into the other $t : \mathcal{L}_L \to \mathcal{L}_{L'}$.
- An *aggregator* is then a function $F : J(\mathcal{X}_L)^n \to J(\mathcal{X}'_{L'})$ such that F is the composition of an aggregator $F' : J(\mathcal{X}_L)^n \to \mathcal{P}(\mathcal{X}_L)$ with a function $T : \mathcal{P}(\mathcal{X}_L) \to \mathcal{P}(\mathcal{X}'_{L'})$ defined by means of t:
  for $J \subset \mathcal{X}_L$, $T(J) = \{t(\phi) \mid \phi \in J\} \subset \mathcal{X}'_{L'}$.
  Thus, we have that $F(\mathbf{J}) = T(F'(\mathbf{J})) \subseteq \mathcal{X}_{\mathbf{L}'}$.
- For example, the majority rule $M : J(\mathcal{X}_L)^n \to J(\mathcal{X}'_{L'})$ is defined as follows. Let $N_\phi = \{i \mid \phi \in J_i\}$, define $M' : J(\mathcal{X}_L)^n \to \mathcal{P}(\mathcal{X}_L)$ as $M'(\mathbf{J}) = \{\phi \in \mathcal{X}_L \mid |N_\phi| > n/2\}$
- Then, given a translation t, $M(\mathbf{J}) = T(M'(\mathbf{J}))$.

Substructural reasoning

- We shall discuss the following logics that are obtained by restricting the language of LL.

MLL	$\neg, \otimes, \parr$
ALL	$\neg, \&, \oplus$
MALL	$\neg, \otimes, \parr, \&, \oplus$

- Moreover, we shall discuss the logics: L + (W) and L + (C) that are obtained by adding weakening or contraction to the logic L.
- Note that CL is equivalent to assuming MALL + (W) and (C).

Group reasoning

- In order to investigate collective rationality for a number of logics, we introduce the following notion of group reasoning. We say that N_ϕ is a *winning coalition* wrt an aggregation procedure F, and we denote it W_ϕ, iff $\phi \in F(\mathbf{J})$.
- We assume a distinguished set of propositional atoms $i_1, \dots, i_n$ one for each agent in N.
- We model group reasoning in a given logic L as follows. We add to the language of L the set of atoms $i_1, \dots, i_n$. We define non-logical (or proper axioms) $W_\phi \vdash \phi$ for any $\phi \in F(\mathbf{J})$.

Definition 1. (Group reasoning)

We say that the group infers a formula $\phi \in \mathcal{L}_L$ according to L iff, for some sequence of $W_j \vdash \phi_j$, there is a proof $W_1, ..., W_m \vdash_L \phi$.

Thus, the notion of group reasoning depends on the logic L as well as on the aggregation rule that defines the non-logical axioms.

Consistency of group reasoning

Definition 2 (Consistency)

We say that group reasoning is *consistent* wrt L iff the sequent $W_1, \ldots, W_m \vdash_L \emptyset$ is not derivable in L for any sequence of winning coalition.

For sound and complete calculi, our notion of group consistency corresponds to the standard model-theoretic view of consistency of a set of judgments J (i.e. there exists a valuation that makes the formulas in J true) as follows.

Fact 1

Group reasoning wrt L is consistent iff the set $J = \{\phi \mid \text{there are } W_1, \ldots, W_m \text{ s.t. } W_1, \ldots, W_m \vdash_L \phi\}$ has a model.

Safety of agendas

We introduce the following definition of *safety* of an agenda.

Definition 3 (Safety)

We say that an agenda $\mathcal{X}_L$ is *safe* for a class of aggregators F wrt the logic L iff group reasoning wrt L is consistent for any aggregator in F.

The majority rule leads to inconsistency wrt classical logic iff the agenda $\mathcal{X}_{CL}$ violates the so called *median property*: every minimally inconsistent set $Y \subseteq \mathcal{X}_{CL}$ has size at most 2.
E.g. $\{a, b, \neg(a \wedge b)\}$ violates the median property.
By Fact 1, our definition of group reasoning allows for rephrasing the standard JA results

Theorem 1

An agenda $\mathcal{X}_{CL}$ is safe for the majority rule $M : J(\mathcal{X}_{CL})^n \rightarrow J(\mathcal{X}_{CL})$ wrt CL iff $\mathcal{X}_{CL}$ satisfies the median property.

Majority and substructural reasoning: MLL

We consider now the majority rule defined on multiplicative agendas $M : J(\mathcal{X}_{MLL})^n \to J(\mathcal{X}_{MLL})$.
In this case, no new result: the median property characterises again safe agendas.

Theorem 2

An agenda $\mathcal{X}_{MLL}$ is safe for $M : J(\mathcal{X}_{MLL})^n \to J(\mathcal{X}_{MLL})$ iff $\mathcal{X}_{MLL}$ satisfies the median property.

Take a profile with three individuals that provides proper axioms $\{i_1, i_2\} \vdash A$, $\{i_1, i_3\} \vdash B$ and $\{i_2, i_3\} \vdash \neg(A \otimes B)$.

Majority and substructural reasoning: ALL

Take the majority rule defined on additive agendas $M : J(\mathcal{X}_{ALL})^n \to J(\mathcal{X}_{ALL})$. We can now state an interesting possibility result. The key property is the following:

Property 1 ($\mathcal{F}$)

In additive linear logic (ALL) every provable sequent contains exactly two formulas (e.g. $A \vdash B$).

- Since every proof starts with axioms $A \vdash A$, and additive rules do not add any new proposition, every provable sequent contains two formulas of ALL.
- This entails that there are no minimal inconsistent sets of size bigger than two in ALL (if J is inconsistent in ALL, then $J \vdash_{ALL} \emptyset$). Thus, every ALL agenda is safe for M.

Theorem 3 (Safety of ALL)

Any agenda $\mathcal{X}_{ALL}$ is safe for the majority rule $M : J(\mathcal{X}_{ALL})^n \to J(\mathcal{X}_{ALL})$ wrt ALL.

Majority and substructural reasoning: ALL + W

The same language of ALL is not safe, if we add more reasoning power. If we add weakening (W) to ALL, then there are agendas that are no longer safe for majority.

Proposition 1

Agendas $\mathcal{X}$ in ALL are not safe for majority rule $M : J(\mathcal{X}_{ALL})^n \to J(\mathcal{X}_{ALL})$ wrt ALL + (W).

Take a profile such that $W_1 \vdash a$, $W_2 \vdash b$ and $W_3 \vdash \neg(a \,\&\, b)$. In ALL + (W), we have the following proof.

$$\dfrac{\dfrac{\dfrac{W_1 \vdash a}{W_1, W_2 \vdash a}\,\mathrm{W} \qquad \dfrac{W_2 \vdash b}{W_1, W_2 \vdash b}\,\mathrm{W}}{W_1, W_2 \vdash a \,\&\, b} \qquad W_3 \vdash \neg(a \,\&\, b)}{W_1, W_2, W_3 \vdash \emptyset}$$

Note that adding contraction (ALL + C) does not affect the possibility result.

Individual and collective rationality

- We focus on the case in which individuals reason in CL and we will evaluate group reasoning wrt fragments of LL.
- We define the *additive translation* of CL into LL $\text{ADD} : \mathcal{L}_{CL} \to \mathcal{L}_{LL}$.
  For a atomic, $\text{ADD}(a) = a$;
  for A in $\mathcal{L}_{CL}$, $\text{ADD}(\neg A) = \neg(\text{ADD}(A))$, $\text{ADD}(A \wedge B) = \text{ADD}(A) \,\&\, \text{ADD}(B)$, $\text{ADD}(A \vee B) = \text{ADD}(A) \oplus \text{ADD}(B)$.
- We know that every agenda in ALL is safe for the majority rule. Thus, by using ADD, we know that agendas in CL are safe for $M : J(\mathcal{X}_{CL})^n \to J(\mathcal{X}_{ALL})$ with $M(\mathbf{J}) = \text{ADD}(M'(\mathbf{J}))$ wrt group reasoning in ALL.

Thus, by Theorem 3 we can prove:

Corollary 1

Any agenda $\mathcal{X}_{CL}$ is safe for the majority rule $M : J(\mathcal{X}_{CL})^n \to J(\mathcal{X}_{ALL})$ wrt ALL.

Individual and collective rationality

- We can extend the previous result, by showing that the majority is always consistent wrt reasoning in LL, provided the additive translation that we have introduced.
- Define the *deductive closure* of a set X wrt to L, $cl_L(X)$, as the set $\{A \mid X \vdash_L A\}$.

Corollary 2

Any agenda $\mathcal{X}_{CL}$ is safe for the majority rule wrt $cl_{MALL}(M(\mathbf{J}))$.

Remark

- The additive translation suggests the following interpretation of linear logic connectives. The distinction between additive and multiplicative connectives allows for separating propositions that are collectively accepted (or inferred) by means of a single winning coalition and proposition that are derived by combining coalitions.

$$\dfrac{\dfrac{\{i_1, i_2\} \vdash a \qquad \{i_1, i_3\} \vdash b}{\{i_1, i_2\}, \{i_1, i_3\} \vdash a \otimes b}\, R\otimes \qquad \{i_2, i_3\} \vdash \neg(a \,\&\, b)}{\cdots}$$

- Recall that in the famous real case of doctrinal paradox judges were reasoning by applying the norm *the confession has been forced and the confession is relevant iff the process has to be redone*. We can express the norm in two ways:

$$c \,\&\, r \leftrightarrow p$$

$$c \otimes r \leftrightarrow p$$

The additive conjunction provide a narrower form of reasoning, namely it requires that the *same* coalition accepts both premises.

Conclusion I

- We have proposed a proof-theoretical analysis of inconsistency in judgment aggregation.
- In particular, we have discussed JA with respect to substructural reasoning and we have seen that the majority rule is consistent wrt ALL, whereas it is enough to add weakening to obtain again discursive dilemmas.
- Moreover, we have defined aggregators that associate possibly heterogeneous notions of individual and collective rationality. In particular, we have shown that the aggregation of classically rational judgments is consistent wrt LL, provided our additive translation. We have suggested an intuitive interpretation of the meaning of linear logic connectives in terms of coalitional reasoning.

Conclusions II

- The treatment that we have presented for the majority rule can be extended to classes of function. In particular, it is known that the majority rule is characterized by the axioms: (A), (I), (M), and (WR). The possibility result provided wrt ALL does not extend to classes of aggregators that are obtained by weakening the axiomatization of the majority rule.
  (E.g. the class of uniform quota rules are characterized by (A), (N), (M) and (I)).
- Future work shall compare the possibility results that have been achieved here proof-theoretically with the semantic definitions of consequence relations. In particular, it is interesting to compare our results with the general logic approach in (Dietrich, 2007).
- Moreover, it is interesting to study preference aggregation with constraints (e.g. transitivity) expressed in linear logic (additive vs multiplicative formulation).