Isomorphism classes of Drinfeld modules over finite fields

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Set-up

We fix some notation:

- $A = \mathbb{F}_q[T]$, $F = \mathbb{F}_q(T)$.
- $\mathfrak{p} \trianglelefteq A$ is a prime of degree d, monic generator denoted by $\mathfrak{p}$.
- $k \cong \mathbb{F}_{q^n}$ is a finite extension of $A/\mathfrak{p} = \mathbb{F}_{\mathfrak{p}} \cong \mathbb{F}_{q^d}$.
- $\gamma\colon A \to A/\mathfrak{p} \hookrightarrow k$ is the A-field structure on k.
- $\pi = \tau^n$ is the Frobenius endomorphism.

Let $\phi : A \to k\{\tau\}$ be a Drinfeld module over k of rank r, with $\mathcal{E} := \mathrm{End}_k(\phi)$ and $D := \mathcal{E} \otimes_A F = \mathrm{End}_k^0(\phi)$.

$$\begin{tikzcd}[arrows=dash]
 & D \arrow[d] & \\
 & \tilde{F} = F(\pi) \arrow[dl] \arrow[dr] & \\
F & & K = \mathbb{F}_q(\pi)
\end{tikzcd}$$

Guiding questions

The isomorphism class of D (or $\tilde{F}$), or equivalently the minimal polynomial of π over F, determines an **isogeny class** of Drinfeld modules over k.
We will consider the case where $D = \tilde{F}$ is commutative.

Important open problem

Describe, determine, and count the isomorphism classes within a fixed isogeny class.

- Brute force results for $r = 2, 3$. [Assong].
- Description of endomorphism rings due to Angles, Garai-Papikian, Kuhn-Pink, and others.
- Related to calculating zeta functions of Drinfeld modular varieties.

Isogenies, subgroups, lattices, ideals [Laumon]

Let $u : \phi \to \psi$ be an isogeny of Drinfeld modules of rank r over k. The kernel of $u \in k\{\tau\}$ is a finite group scheme G_u in A-modules.

Let $H_\mathfrak{p}$ denote the Dieudonné module and $T_\mathfrak{l}$ the Tate module. Via injective maps $u_\mathfrak{p} : H_\mathfrak{p}(\psi) \hookrightarrow H_\mathfrak{p}(\phi)$ and $u_\mathfrak{l} : T_\mathfrak{l}(\phi) \hookrightarrow T_\mathfrak{l}(\psi)$ for $\mathfrak{l} \neq \mathfrak{p}$, it yields sublattices $M_\mathfrak{p} := u_\mathfrak{p}(H_\mathfrak{p}(\psi)) \subseteq H_\mathfrak{p}(\phi)$ and $M_\mathfrak{l} := \mathrm{Hom}(u_\mathfrak{l}^{-1} T_\mathfrak{l}(\psi), A_\mathfrak{l}) \subseteq \mathrm{Hom}(T_\mathfrak{l}(\phi), A_\mathfrak{l}) =: H_\mathfrak{l}(\phi)$ for $\mathfrak{l} \neq \mathfrak{p}$, and hence a sublattice $M := \prod_\mathfrak{l} M_\mathfrak{l} \subseteq \prod_\mathfrak{l} H_\mathfrak{l}(\phi) =: \mathbb{H}(\phi)$. By construction, $G_u \simeq \prod_{\mathfrak{l} \neq \mathfrak{p}} H_\mathfrak{l}(\phi)/M_\mathfrak{l} \times H_\mathfrak{p}(\phi)/M_\mathfrak{p} = \mathbb{H}(\phi)/M$.

For an ideal $I \trianglelefteq \mathcal{E}$, we have $k\{\tau\}I = k\{\tau\}u_I$ for some $u_I \in k\{\tau\}$. The sublattice corresponding to u_I is $I\mathbb{H}(\phi) = \prod_\mathfrak{l} IH_\mathfrak{l}(\phi)$, since $\ker(u_I) = \phi[I] = \cap_{\alpha \in I} \ker(\alpha)$.

Ideal action on isomorphism classes [Hayes]

Recall $\phi : A \to k\{\tau\}$ is a Drinfeld module with $\mathcal{E} := \mathrm{End}_k(\phi)$.
For an ideal $I \trianglelefteq \mathcal{E}$, again write $k\{\tau\}I = k\{\tau\}u_I$ with $u_I \in k\{\tau\}$.

A Drinfeld module over k is determined by its value at T.
Setting $\psi_T = u_I \phi_T u_I^{-1}$ determines a Drinfeld module ψ over k, isogenous to ϕ via $u_I : \phi \to \psi$. We write $\psi = I * \phi$.

Lemma

The map $I \mapsto I * \phi$ determines an action of the monoid of fractional ideals of $\mathcal{E}$ up to linear equivalence on the set of isomorphism classes in the isogeny class of ϕ.

When is this action free? When is it transitive?

Kernel ideals

Let $I \trianglelefteq \mathcal{E} := \mathrm{End}_k(\phi) = D \cap k\{\tau\}$ be an ideal.

Definition

The ideal I is a **kernel ideal** if any of the following equivalent properties holds:

1. $I = (k\{\tau\}I) \cap D$. (Generally $\subseteq$.) [Yu]
2. $I = \mathrm{Ann}_{\mathcal{E}}(\phi[I])$. (Generally $\subseteq$.)
3. For any $J \trianglelefteq \mathcal{E}$, we have $J\mathbb{H}(\phi) \subseteq I\mathbb{H}(\phi) \Rightarrow J \subseteq I$. ($\Leftarrow$ holds.)

Lemma

Upon restricting to kernel ideals, the ideal action $I \mapsto I * \phi$ is free.

Lemma

Every ideal is a kernel ideal when $\mathcal{E}$ is maximal, or when $\mathcal{E}$ is Gorenstein, e.g., when $\mathcal{E} = A[\pi]$.

Endomorphism rings (under the ideal action)

Fix a commutative endomorphism algebra D, i.e., an isogeny class. The endomorphism ring $\mathcal{E}$ of a Drinfeld module ϕ in the isogeny class is an order in D containing the minimal order $A[\pi]$.
For $I \trianglelefteq \mathcal{E}$, let $(I : I) = \{g \in D : Ig \subseteq I\}$ be its order.
Write $k\{\tau\}I = k\{\tau\}u_I$ as before.

Lemma, cf. [Yu] and [Waterhouse]

For any $I \trianglelefteq \mathcal{E}$, we have $\mathrm{End}_k(I * \phi) \supseteq u_I(I : I)u_I^{-1} \simeq (I : I)$.
Equality holds when I is a kernel ideal.

Since $\mathcal{E} \subseteq (I : I)$, "endomorphism rings grow under ideal action".
For transitivity of $I \mapsto I * \phi$, every occurring endomorphism ring in the isogeny class should be an overorder of $\mathcal{E}$.
When does the minimal order $A[\pi]$ occur as endomorphism ring?

Local maximality of $A[\pi]$

$$\begin{tikzcd}[column sep=small] & D = \tilde{F} = F(\pi) \arrow[dl, dash] \arrow[dr, dash] & \\ F & & K = \mathbb{F}_q(\pi) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \tilde{\mathfrak{p}} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathfrak{p} & & (\pi) \end{tikzcd}$$

Definition, cf. [Angles]

Let $B_{\tilde{\mathfrak{p}}}$ be the ring of integers of $\tilde{F}_{\tilde{\mathfrak{p}}} := \tilde{F} \otimes_K \mathbb{F}_q((\pi))$ and write $A[\pi]_{\tilde{\mathfrak{p}}} := A[\pi] \otimes_{\mathbb{F}_q[\pi]} \mathbb{F}_q[[\pi]]$. Then $A[\pi]$ is **locally maximal** at π if $A[\pi]_{\tilde{\mathfrak{p}}} = B_{\tilde{\mathfrak{p}}}$.

Theorem

Recall $\deg(\mathfrak{p}) = d$ and $k \simeq \mathbb{F}_{q^n}$. Let H be the height of ϕ.

Then $\left\lceil \frac{n}{H \cdot d} \right\rceil \leq \frac{[\tilde{F}:K]}{d}$, with equality $\Leftrightarrow$ $A[\pi]$ is locally maximal at π.

Hence, $A[\pi]$ is locally maximal at $\pi \Leftrightarrow \phi$ is ordinary or $k = \mathbb{F}_{\mathfrak{p}}$.

$A[\pi]$ as an endomorphism ring

Fix a commutative endomorphism algebra D, i.e., an isogeny class.

Lemma

For any overorder R of $A[\pi]$, there exists a Drinfeld module ϕ in the isogeny class such that $\mathrm{End}_k(\phi)_{\mathfrak{l}} = R_{\mathfrak{l}}$ for all $\mathfrak{l} \neq \mathfrak{p}$.

At $\mathfrak{p}$, i.e., at π, any endomorphism ring is locally maximal. [Yu]
By the Theorem, $A[\pi]$ is locally maximal at π if and only if ϕ is ordinary or $k = \mathbb{F}_{\mathfrak{p}}$.

Corollary

$A[\pi]$ occurs as an endomorphism ring if and only if it is locally maximal at π, if and only if the isogeny class is ordinary or $k = \mathbb{F}_{\mathfrak{p}}$. So does any overorder of $A[\pi]$.

Main result

Theorem

Suppose that $\mathcal{E} := \mathrm{End}_k(\phi) = A[\pi]$. Then the action $I \mapsto I * \phi$ of the monoid of fractional ideals of $A[\pi]$ is free and transitive on the isomorphism classes in the isogeny class of ϕ.
In other words, all isomorphism classes in the isogeny class of ϕ are of the form $I * \phi$ for some $A[\pi]$-ideal I.

- If $\mathcal{E} = A[\pi]$ then ϕ is ordinary or $k = \mathbb{F}_{\mathfrak{p}}$.
- For the Gorenstein order $A[\pi]$, every ideal is a kernel ideal.
- Kernel ideals act freely.
- Kernel ideals of $A[\pi]$ act transitively on isomorphism classes whose endomorphism ring is an overorder of $A[\pi]$, i.e., on all isomorphism classes.

Comparison with abelian varieties A over $\mathbb{F}_q$

Also in this case it is an interesting open problem to describe the isomorphism classes within a fixed isogeny class, determined by the isomorphism class of $E = \mathrm{End}^0(A)$, or equivalently by the minimal polynomial of their Frobenius endomorphism π.

Deligne proved that the functor $A \mapsto H_1(\tilde{A} \otimes \mathbb{C})$ induces an equivalence of categories between **ordinary** abelian varieties over $\mathbb{F}_q$ and certain $\mathbb{Z}[\pi, \bar{\pi}]$-modules, via the canonical lifting $\tilde{A}$ of A to characteristic zero. These $\tilde{A}$ have a complex uniformisation.

Centeleghe-Stix proved that the functor $A \mapsto \mathrm{Hom}_{\mathbb{F}_p}(A, A_w)$ induces an equivalence of categories between abelian varieties **over** $\mathbb{F}_p$ and certain $\mathbb{Z}[\pi, \bar{\pi}]$-modules, where A_w has minimal endomorphism ring.