Mass formulae for supersingular abelian varieties

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Curves over finite fields: past, present and future

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Moduli space $\mathcal{A}_g$

Let k be an algebraically closed field of characteristic p.

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For $X \in \mathcal{A}_g(k)$, consider its p-divisble group $X[p^\infty]$.
The isogeny class of $X[p^\infty]$ uniquely determines a Newton polygon.
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The isogeny class of $X[p^\infty]$ also determines the p-RANK f of X:
$|X[p](k)| = p^f$, so $0 \leq f \leq g$.
$\Rightarrow$ ***p*-rank stratification** of $\mathcal{A}_g$.

Moduli space $\mathcal{S}_g$

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1. $X \in \mathcal{A}_g(k)$ is SUPERSINGULAR if $X \sim E^g$ with $E[p](k) = 0$.
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- Every component of $\mathcal{S}_g$ has dimension $\lfloor \frac{g^2}{4} \rfloor$.

The a-number stratification

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Let $X \in \mathcal{A}_g(k)$. Its a-NUMBER is $a(X) := \dim_k \mathrm{Hom}(\alpha_p, X)$.
It depends on the isomorphism class of $X[p]$.

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- Every component of $\mathcal{S}_g(a)$ has dimension $\lfloor \frac{g^2-a^2+1}{4} \rfloor$.
- $a(X) = g \Leftrightarrow X$ is SUPERSPECIAL, i.e., $X \simeq E^g$.
  The superspecial stratum $S_g(g)$ is zero-dimensional.

The Ekedahl-Oort stratification

For $X \in \mathcal{A}_g(k)$, consider its p-torsion $X[p]$.
Its isomorphism class is classified by an element of the Weyl group W_g of Sp_{2g}, or equivalently by an ELEMENTARY SEQUENCE φ.
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- Ekedahl-Oort stratification refines the p-rank stratification.
- Also consider Ekedahl-Oort stratification $\coprod_\varphi (\mathcal{S}_\varphi \cap \mathcal{S}_g)$ of $\mathcal{S}_g$.
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  Combinatorial criterion determines when $\mathcal{S}_\varphi \subseteq \mathcal{S}_g$.
  These strata are reducible; all other strata are irreducible.
- The a-number is constant on Ekedahl-Oort strata.
  $\Rightarrow \mathcal{S}_g(a) = \coprod_\varphi (\mathcal{S}_\varphi \cap \mathcal{S}_g)$.

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$$\Lambda_x = \{(X, \lambda) \in \mathcal{S}_g(k) : (X, \lambda)[p^\infty] \simeq (X_0, \lambda_0)[p^\infty]\}.$$

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- Let $G_x/\mathbb{Z}$ be the automorphism group scheme, such that

$$G_x(R) = \{h \in (\mathrm{End}(X_0) \otimes_{\mathbb{Z}} R)^\times : h'h = 1\}$$

for any commutative ring R. Then there is a bijection

$$\Lambda_x \simeq G_x(\mathbb{Q}) \backslash G_x(\mathbb{A}_f) / G_x(\widehat{\mathbb{Z}}).$$

A finer stratification?

$$\Lambda_x = \{(X,\lambda) \in \mathcal{S}_g(k) : (X,\lambda)[p^\infty] \simeq (X_0,\lambda_0)[p^\infty]\}.$$

Goal

For any $x \in \mathcal{S}_g$, compute the MASS

$$\mathrm{Mass}(\Lambda_x) = \sum_{x' \in \Lambda_x} |\mathrm{Aut}(x')|^{-1}.$$

N.B. $\mathrm{Mass}(\Lambda_x) = \mathrm{vol}(G_x(\mathbb{Q}) \backslash G_x(\mathbb{A}_f)) = \mathrm{Mass}(G_x, G_x(\widehat{\mathbb{Z}}))$.

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From now on, we work with $g = 3$!

How do we describe $\mathcal{S}_3$?

Let $E/\mathbb{F}_{p^2}$ be a supersingular elliptic curve with $\pi_E = -p$.
Let μ be any principal polarisation of E^3.

Definition

A POLARISED FLAG TYPE QUOTIENT (PFTQ) WITH RESPECT TO μ is a chain

$$(E^3, p\mu) =: (Y_2, \lambda_2) \xrightarrow{\rho_2} (Y_1, \lambda_1) \xrightarrow{\rho_1} (Y_0, \lambda_0)$$

such that $\ker(\rho_1) \simeq \alpha_p$, $\ker(\rho_2) \simeq \alpha_p^2$, and $\ker(\lambda_i) \subseteq \ker(V^j \circ F^{i-j})$ for $0 \leq i \leq 2$ and $0 \leq j \leq \lfloor i/2 \rfloor$.

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Let $\mathcal{P}_\mu$ be the moduli space of PFTQ's.
It is a two-dimensional geometrically irreducible scheme over $\mathbb{F}_{p^2}$.

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It follows that $(Y_0, \lambda_0) \in \mathcal{S}_3$, so there is a projection map

$$\begin{aligned} \mathrm{pr}_0 : \mathcal{P}_\mu &\to \mathcal{S}_3 \\ (Y_2 \to Y_1 \to Y_0) &\mapsto (Y_0, \lambda_0) \end{aligned}$$

such that $\coprod_\mu \mathcal{P}_\mu \to \mathcal{S}_3$ is surjective and generically finite.

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Slogan

Each $\mathcal{P}_\mu$ approximates a geom. irreducible component of $\mathcal{S}_3$.

How do we describe $\mathcal{P}_\mu$?

Let $C : t_1^{p+1} + t_2^{p+1} + t_3^{p+1} = 0$ be a Fermat curve in $\mathbb{P}^2$.
It has genus $p(p-1)/2$ and admits a left action by $U_3(\mathbb{F}_p)$.

Then $\pi : \mathcal{P}_\mu \simeq \mathbb{P}_C(\mathcal{O}(-1) \oplus \mathcal{O}(1)) \to C$ is a $\mathbb{P}^1$-bundle.
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Upshot

For each (X, λ) there exist a μ and a $y \in \mathcal{P}_\mu$ such that $\mathrm{pr}_0(y) = [(X, \lambda)]$.
This y is uniquely characterised by a pair (t, u) with $t = (t_1 : t_2 : t_3) \in C(k)$ and $u = (u_1 : u_2) \in \pi^{-1}(t) \simeq \mathbb{P}^1_t(k)$.

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Recall that X/k has a-number $a(X) = \dim_k \mathrm{Hom}(\alpha_p, X)$.
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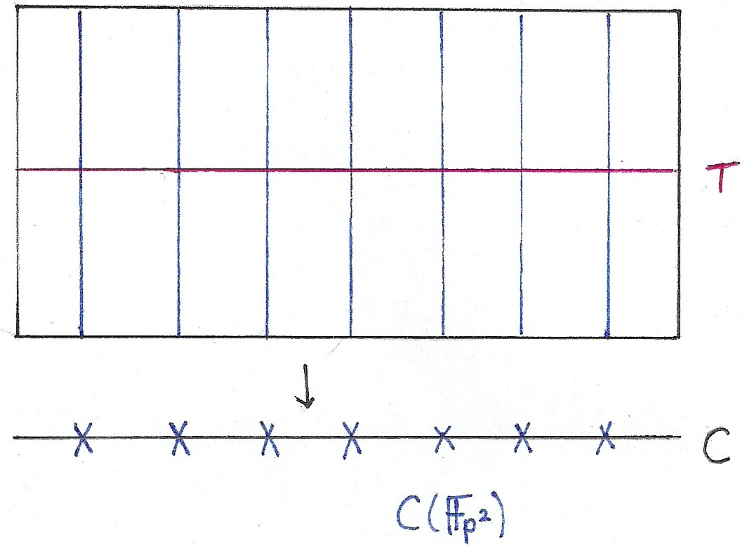
- For $y \in \mathcal{P}_\mu$, we have $a(y) = 1 \Leftrightarrow y \notin T$ and $\pi(y) \notin C(\mathbb{F}_{p^2})$.

The structure of $\mathcal{P}_\mu$: a picture



Using PFTQ's to construct minimal isogenies

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Idea

Construct the minimal isogeny for X from its corresponding PFTQ

$$Y_2 \xrightarrow{\rho_2} Y_1 \xrightarrow{\rho_1} Y_0 = X.$$

(If $Y_2 \to Y_1 \to Y_0$ is a PFTQ, then Y_2 is superspecial!)

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- If $a(X) = 3$ then X is superspecial and $\varphi = \mathrm{id}$.
- If $a(X) = 2$, then $a(Y_1) = 3$ and $\varphi = \rho_1$ of degree p.
- If $a(X) = 1$, then $\varphi = \rho_1 \circ \rho_2$ of degree p^3.

What is a mass formula?

Goal

Compute $\mathrm{Mass}(\Lambda_x) = \sum_{x' \in \Lambda_x} |\mathrm{Aut}(x')|^{-1}$ for any $x \in \mathcal{S}_3$.

A mass formula computes an expression for the mass.

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Eichler-Deuring mass formula

Let $S = \{$ supersingular elliptic curves over $\overline{\mathbb{F}}_p\}/\simeq$. Then

$$\mathrm{Mass}(S) = \sum_{s \in S} \frac{1}{|\mathrm{Aut}(s)|} = \frac{p-1}{24}.$$

From minimal isogenies to masses

Let $x = (X, \lambda)$ be supersingular and $\varphi : Y \to X$ a minimal isogeny. Write $\tilde{x} = (Y, \varphi^*\lambda)$. Recall automorphism group scheme G_x.

Through φ, we may view both $G_{\tilde{x}}(\widehat{\mathbb{Z}})$ and $\varphi^* G_x(\widehat{\mathbb{Z}})$ as open compact subgroups of $G_{\tilde{x}}(\mathbb{A}_f)$, which differ only at p.

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Lemma

$$\begin{aligned}\mathrm{Mass}(\Lambda_x) &= \frac{[G_{\tilde{x}}(\widehat{\mathbb{Z}}) : G_{\tilde{x}}(\widehat{\mathbb{Z}}) \cap \varphi^* G_x(\widehat{\mathbb{Z}})]}{[\varphi^* G_x(\widehat{\mathbb{Z}}) : G_{\tilde{x}}(\widehat{\mathbb{Z}}) \cap \varphi^* G_x(\widehat{\mathbb{Z}})]} \cdot \mathrm{Mass}(\Lambda_{\tilde{x}}) \\ &= [\mathrm{Aut}((Y, \phi^*\lambda)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] \cdot \mathrm{Mass}(\Lambda_{\tilde{x}}).\end{aligned}$$

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$$= [\mathrm{Aut}((Y, \phi^* \lambda)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] \cdot \mathrm{Mass}(\Lambda_{\tilde{x}}).$$

So we can compare any supersingular mass to a superspecial mass.

From minimal isogenies to masses

Moreover, the superspecial masses are known in any dimension!

Lemma [Ekedahl, Harashita, Hashimoto, Ibukiyama, Yu]

Let $\tilde{x} = (Y, \lambda)$ be a superspecial abelian threefold.

- If λ is a principal polarisation, then
$$\mathrm{Mass}(\Lambda_{\tilde{x}}) = \frac{(p-1)(p^2+1)(p^3-1)}{2^{10} \cdot 3^4 \cdot 5 \cdot 7}.$$
- If $\ker(\lambda) \simeq \alpha_p \times \alpha_p$, then
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It remains to compute $[\mathrm{Aut}((Y, \phi^*\lambda)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])]$.

The case $a(X) = 2$

Let $x = (X, \lambda) \in \mathcal{S}_3$ such that $a(X) = 2$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2})$.
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There are reduction maps

$$\mathrm{Aut}((Y_1, \lambda_1)[p^\infty]) \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_{p^2})$$
$$\mathrm{Aut}((X, \lambda)[p^\infty]) \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_{p^2}) \cap \mathrm{End}(u)^\times,$$

where

$$\mathrm{End}(u) = \{g \in M_2(\mathbb{F}_{p^2}) : g\cdot u \subseteq k\cdot u\} \simeq \begin{cases} \mathbb{F}_{p^4} \text{ if } u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ \mathbb{F}_{p^2} \text{ if } u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^4}). \end{cases}$$

The case $a(X) = 2$

Let $x = (X, \lambda) \in \mathcal{S}_3$ such that $a(X) = 2$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2})$.

So $[\mathrm{Aut}((Y_1, \lambda_1)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] =$

$$[\mathrm{SL}_2(\mathbb{F}_{p^2}) : \mathrm{SL}_2(\mathbb{F}_{p^2}) \cap \mathrm{End}(u)^\times] = \begin{cases} p^2(p^2-1) & \text{if } u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ |\mathrm{PSL}_2(\mathbb{F}_{p^2})| & \text{if } u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^4}). \end{cases}$$

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Theorem (K.-Yobuko-Yu)

There are two mass strata in $\mathcal{S}_3(2)$:

$$\mathrm{Mass}(\Lambda_x) = \frac{1}{2^{10} \cdot 3^4 \cdot 5 \cdot 7} \cdot \begin{cases} (p-1)(p^3+1)(p^3-1)(p^4-p^2) & : u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ 2^{-e(p)}(p-1)(p^3+1)(p^3-1)p^2(p^4-1) & : u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^4}). \end{cases}$$

The case $a(X) = 1$

Let $x = (X, \lambda) \in \mathcal{S}_3$ such that $a(X) = 1$.
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Let $D_p = \mathbb{Q}_{p^2}[\Pi]$ be the division quaternion algebra over $\mathbb{Q}_p$, and let $\mathcal{O}_{D_p}$ its maximal order. (We have $\Pi^2 = -p$.)

- $G_2 := \mathrm{Aut}((Y_2, \lambda_2)[p^\infty]) \simeq \{A \in \mathrm{GL}_3(\mathcal{O}_{D_p}) : A^* A = \mathbb{I}_3\}$.
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Reducing modulo p we obtain $\overline{G}_2$ and $\overline{G}$, where:

- $\overline{G}_2 = \{A + B\Pi \in \mathrm{GL}_3(\mathbb{F}_{p^2}[\Pi]) : A^* A = \mathbb{I}_3, B^T A^* = A^{*T} B\}$, so $|\overline{G}_2| = |U_3(\mathbb{F}_p)| \cdot |S_3(\mathbb{F}_{p^2})| = p^{15}(p+1)(p^2-1)(p^3+1)$;
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Moreover,

- $[\mathrm{Aut}((Y_2, \lambda_2)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] = [G_2 : G] = [\overline{G}_2 : \overline{G}]$.

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- $$\overline{G} \simeq \left\{ \begin{pmatrix} A & 0 \\ SA & A^{(p)} \end{pmatrix} : A \in U_3(\mathbb{F}_p), A \cdot t = \alpha \cdot t, \; S \in S_3(\mathbb{F}_{p^2}), \psi_t(S) = u_2 u_1^{-1}(1 - \alpha^{p^3-1}) \right\},$$
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The images of ψ_t for varying t define a divisor $D \subseteq C^0 \times \mathbb{P}^1$.
For $t \in C^0(k)$, let $d(t) = \dim_{\mathbb{F}_{p^2}}(\mathrm{Im}(\psi_t))$ and $D_t = \pi^{-1}(t) \cap D$.
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- $$|\overline{G}| = \begin{cases} 2^{e(p)} p^{2(6-d(t))} & \text{if } u \notin D_t; \\ (p+1) p^{2(6-d(t))} & \text{if } u \in D_t \text{ and } t \notin C(\mathbb{F}_{p^6}); \\ (p^3+1) p^6 & \text{if } u \in D_t \text{ and } t \in C(\mathbb{F}_{p^6}). \end{cases}$$

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Theorem (K.-Yobuko-Yu)

There are three mass strata in $\mathcal{S}_3(1)$:

$$\mathrm{Mass}(\Lambda_x) = \frac{p^3}{2^{10} \cdot 3^4 \cdot 5 \cdot 7}.$$

$$\begin{cases} 2^{-e(p)} p^{2d(t)}(p^2 - 1)(p^4 - 1)(p^6 - 1) & : u \notin D_t; \\ p^{2d(t)}(p - 1)(p^4 - 1)(p^6 - 1) & : u \in D_t, t \notin C(\mathbb{F}_{p^6}); \\ p^6(p^2 - 1)(p^3 - 1)(p^4 - 1) & : u \in D_t, t \in C(\mathbb{F}_{p^6}). \end{cases}$$

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Question

What else can we use all these computations for?

Application: Oort's conjecture

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Every generic g-dimensional principally polarised supersingular abelian variety (X, λ) over k of characteristic p has automorphism group $C_2 \simeq \{\pm 1\}$.

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When $g = 3$, Oort's conjecture holds precisely when $p \neq 2$.

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Theorem (K.-Yobuko-Yu)

When $g = 3$, Oort's conjecture holds precisely when $p \neq 2$.

- A *generic* threefold X has $a(X) = 1$.
  Its PFTQ is characterised by $t \in C^0(k)$ and $u \notin D_t$.
- Our computations show for such (X, λ) that

$$\mathrm{Aut}((X, \lambda)) \simeq \begin{cases} C_2^3 & \text{for } p = 2; \\ C_2 & \text{for } p \neq 2. \end{cases}$$