Moduli spaces:
classifying, constructing and counting varieties

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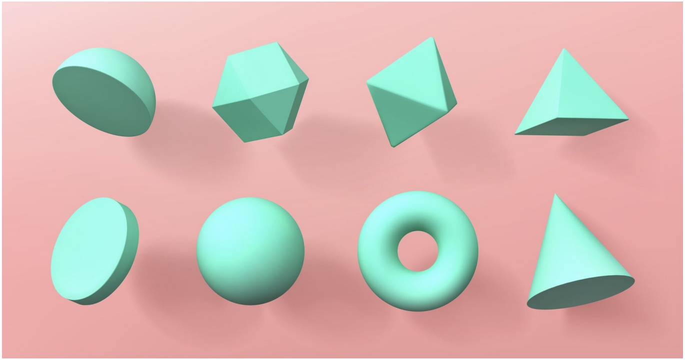
How do we classify geometric objects?



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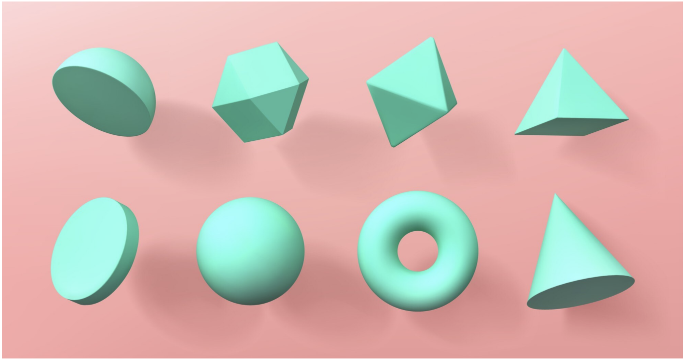

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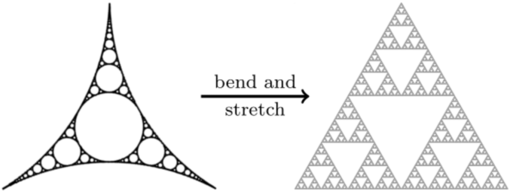
Example: triangles in the real plane ($\mathbb{R}^2$).

Topology: two triangles are the same (= homeomorphic) if you can continuously deform one into the other. So all triangles are the same, and they're also the same as all circles, squares, ...



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Example: triangles in the real plane ($\mathbb{R}^2$) with non-zero area.

Geometry: we no longer allow all ways of deforming.
For example, we may call two triangles the same if they are:

1. Equal (= same vertices with same labelling), or
2. Similar (= congruent up to scaling).

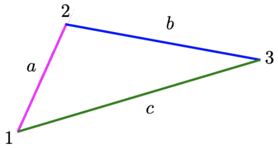


Figure: 1. Equal

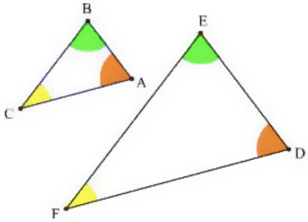


Figure: 2. Similar

How many different triangles are there?[1]

1. The set of all non-equal triangles in $\mathbb{R}^2$ is

$$M_1 = \{(x_1, y_1, x_2, y_2, x_3, y_3) : \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0\}.$$

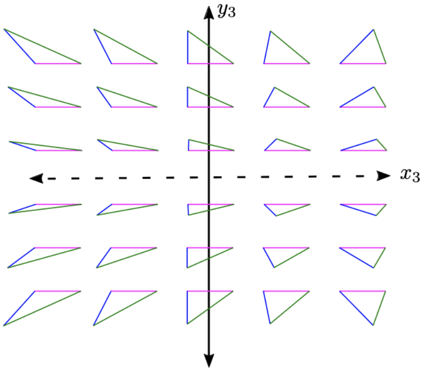


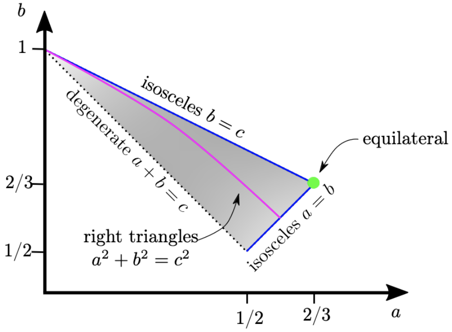
Figure: Slice of M_1 where $(x_1, y_1) = (0, 0), (x_2, y_2) = (1, 0)$.

[1] Jarod Alper, *Stacks and Moduli*, version July 2025.

How many different triangles are there?

2. The set of all non-similar triangles in $\mathbb{R}^2$ is

$$M_2 = \{(a, b, c) : 0 < a \leq b \leq c < a + b, a + b + c = 2\}.$$



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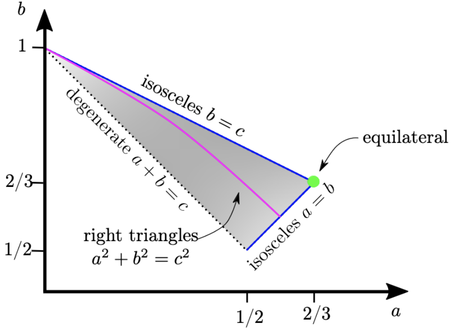

Figure: M_2 with some special triangles marked

Question: What does the set of all non-congruent triangles look like?

What have we just done?

- We chose a suitable notion of “being the same” for our objects.
- We found a description of the set of objects that are not the same.
- Every point in the set is a unique object.

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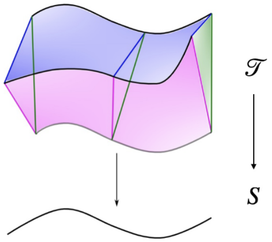

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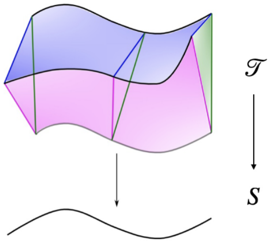

Slogan: In a moduli space every point corresponds with a unique object that is different (= not the same) as all other objects.

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Algebraic geometry is not only about triangles, but about:

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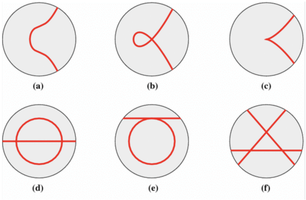
Varieties: Solution sets of polynomial equations,

of a certain *dimension* (="number of equations"):

Curves (dim 1), surfaces (dim 2), ...

Curves have a *genus* (="complexity of the equation"):

Lines (genus 0), **elliptic curves** (genus 1), ...


(a) (b) (c)
(d) (e) (f)

What is an elliptic curve?

Definition (elliptic curve)

An elliptic curve is a curve of genus 1,
given by a Weierstrass equation:

$$E : y^2 + axy + by = x^3 + cx^2 + dx + e$$

for parameters a, b, c, d, e, together with a point $\mathcal{O}$ ("at infinity").

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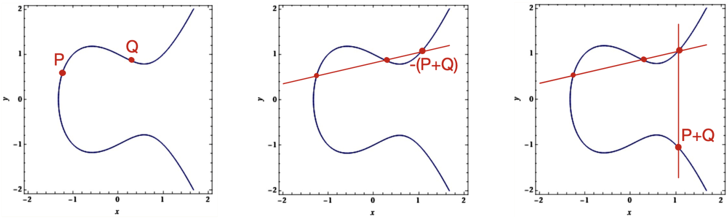
The solutions of the equation, i.e. the points on the curve, can be added together via a geometric process (where $\mathcal{O} = 0$).



When are two elliptic curves the same?

Two elliptic curves

$$E_1 : y^2 + axy + by = x^3 + cx^2 + dx + e$$
$$E_2 : \tilde{y}^2 + f\tilde{x}\tilde{y} + g\tilde{y} = \tilde{x}^3 + h\tilde{x}^2 + i\tilde{x} + j$$

are the same (= *isomorphic*) if we can go from E_1 to E_2 via

$$\boldsymbol{\tilde{x} = u^2 x + r, \quad \tilde{y} = u^3 y + u^2 s x + t},$$

for constants u, r, s, t with $u \neq 0$.

Then we have

$$\begin{aligned} ua &= f + 2s, \\ u^2 c &= h - sf + 3r - s^2, \\ u^3 b &= g + rf + 2t, \\ u^4 d &= i - sg + 2rh - (t + rs)f + 3r^2 - 2st, \\ u^6 e &= j + ri + r^2 h + r^3 - tg - t^2 - rtf. \end{aligned}$$

A family of elliptic curves

After a coordinate change we write the Weierstrass equation as

$$E_\lambda : y^2 = x(x-1)(x-\lambda),$$

for $\lambda \neq 0, 1$.

This is called the **Legendre normal form** of the elliptic curve.

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To find an elliptic curve $\tilde{y}^2 = \tilde{x}(\tilde{x}-1)(\tilde{x}-\mu)$ isomorphic to E_λ we take

$$\tilde{x} = u^2 x + r \text{ en } \tilde{y} = u^3 y,$$

so that

$$x(x-1)(x-\mu) = \left(x + \frac{r}{u^2}\right)\left(x + \frac{r-1}{u^2}\right)\left(x + \frac{r-\lambda}{u^2}\right).$$

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So we have six choices of μ! They are:

$$\lambda,\ 1/\lambda,\ 1-\lambda,\ 1/(1-\lambda),\ \lambda/(\lambda-1),\ (\lambda-1)/\lambda.$$

The j-invariant

We look for another expression in the coefficients of an elliptic curve that describes it uniquely up to isomorphism.

After another coordinate change we simplify the general equation to

$$E : y^2 = x^3 + Ax + B.$$

Definition (j-invariant)

An elliptic curve given by a Weierstrass equation

$$E : y^2 = x^3 + Ax + B$$

with parameters A, B has j-invariant

$$j(E) = 12^3 \frac{4A^3}{4A^3 + 27B^2}.$$

The moduli space of elliptic curves

$E : y^2 = x^3 + Ax + B$ has j-invariant $j(E) = 12^3 \frac{4A^3}{4A^3+27B^2}$.

The j-invariant is constructed such that elliptic curves are isomorphic if and only if they have the same j-invariant.

Moreover, for every value J of the j-invariant there exists a unique elliptic curve with that j-invariant. It is:

$$E_J : y^2 = x^3 - \frac{27J}{4(J-12^3)}x - \frac{27J}{4(J-12^3)}.$$

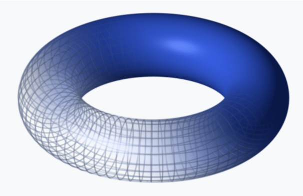
This means:

Conclusion

The line $\{j\}$ is (isomorphic to) the moduli space of elliptic curves. Every point on this line (i.e. every j-value) corresponds to a unique elliptic curve up to isomorphism.

Elliptic curves over $\mathbb{C}$

Working over the complex numbers, we can give an alternative description.
This is because an elliptic curve over $\mathbb{C}$ is a torus:



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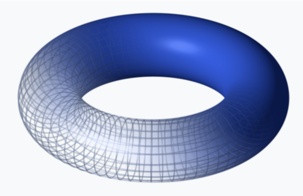

A complex torus can be written as $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ for τ in $\mathbb{C}$.
This means: we call two points c_1, c_2 in $\mathbb{C}$ the same if there exist integers z_1, z_2 such that $c_1 = c_2 + (z_1 + z_2 \cdot \tau)$.

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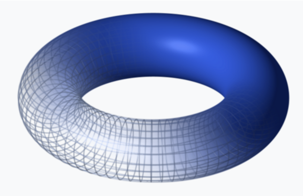

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The points in $\Lambda = \mathbb{Z}+\mathbb{Z}\tau$ describe a lattice.
Tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are the same (isomorphic) $\Leftrightarrow$
there exists c in $\mathbb{C}$ such that $\Lambda_1 = c \cdot \Lambda_2$.
Then Λ_1 and Λ_2 are **homothetic** lattices.

The moduli space of elliptic curves over $\mathbb{C}$

Tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic $\Leftrightarrow \Lambda_1 = c \cdot \Lambda_2$ are homothetic.

To understand the moduli space of elliptic curves (= tori $\mathbb{C}/\Lambda$),
it suffices to describe the moduli space of lattices $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$.

We may choose $\tau = a + bi$ such that $b > 0$.
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But this does not render the choice of τ unique!
One shows that Λ_{τ_1} is homothetic to Λ_{τ_2}, for $\tau_1 = a_1 + b_1 i$ and $\tau_2 = a_2 + b_2 i$ (with $b_1, b_2 > 0$) if and only if

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} Aa_2 + Bb_2 \\ Ca_2 + Db_2 \end{pmatrix} \tag{1}$$

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Conclusion

The moduli space of elliptic curves over $\mathbb{C}$ is (isomorphic to) the set $\{\tau = a + bi : b > 0\}$ up to the equivalence in (1), i.e. to

$$(\mathrm{SL}_2(\mathbb{Z})/\pm\mathrm{Id}_2) \setminus \mathbb{H}.$$

Abelian varieties over finite fields

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I mostly consider abelian varieties which (i.e., whose defining equations) are defined over finite fields.

Definition/notation

Let $\mathbb{F}_q$ be the finite field of cardinality $q = p^r$, where p is a prime.

All elements $x \in \mathbb{F}_q$ satisfy $x^q = x$.

Abelian varieties over finite fields have interesting applications in cryptography and coding theory.

First classification: abelian varieties up to isogeny

When are two abelian varieties X, Y of dimension $g \geq 1$ the same?

First answer: when they are **isogenous**, denoted $X \sim Y$.
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Honda and Tate showed:
$\{$ isogeny classes of abelian varieties over $\mathbb{F}_q$ $\} \leftrightarrow \{$ Weil q-polynomials $\}$

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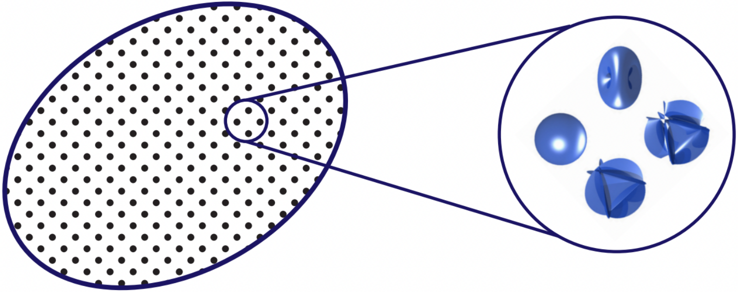
Conclusion

We can parametrise isogeny classes by Weil q- polynomials.
The "space of Weil q-polynomials" is not a nice moduli space, however.

Open problem: describe all abelian varieties in a fixed isogeny class.

Moduli spaces of abelian varieties (over finite fields)

Stronger: X, Y are the same when they are isomorphic, denoted $X \simeq Y$. This yields a nice moduli space, denoted $\mathcal{A}_g$ (in dim g).



Geometry of moduli space $\Rightarrow$ **arithmetic** of families of abelian varieties.

Arithmetic invariants and their induced stratifications

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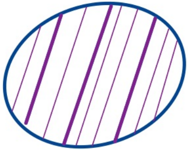
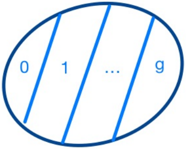
Finite fields have characteristic p, so p-(power) torsion is interesting.
Consider the p-torsion group scheme $X[p]$ and p-divisible group $X[p^\infty]$.

$X[p^\infty]/\sim$: p-rank stratification, refined by Newton stratification





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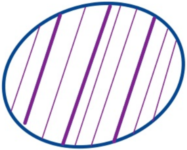
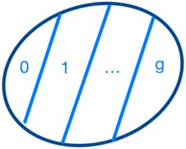
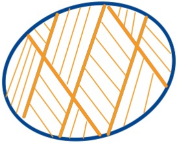
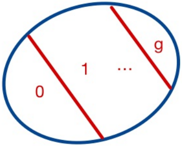




$X[p]/\simeq$: a-number stratification, refined by EO stratification.





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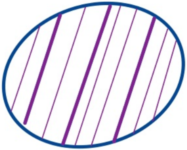

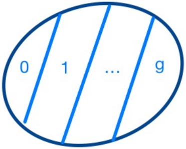

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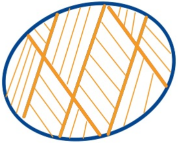

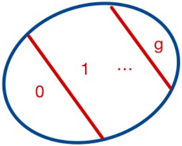

Open problem: How do the Newton and EO strata intersect?

Abelian varieties determined by their p-divisible group

We saw $X[p]/\simeq$ and $X[p^\infty]/\sim$, now $X[p^\infty]/\simeq$.
The **central leaf** passing through abelian variety X_0 is

$$\Lambda_{X_0} = \{X \in \mathcal{A}_g : X[p^\infty] \simeq X_0[p^\infty]\}.$$

Question: For which abelian varieties X_0 do we have $\#\Lambda_{X_0} = 1$?

Theorem (Ibukiyama-K.-Yu)

For a **supersingular** abelian variety X_0 of dimension g, we have $\#\Lambda_{X_0} = 1$ if and only if one of the following holds:

1. $g = 1$ and $p \in \{2, 3, 5, 7, 13\}$;
2. $g = 2$ and $p \in \{2, 3\}$;
3. $g = 3$, $p = 2$, and $a(X_0) \geq 2$.

(Here we work over the algebraically closed field $k = \overline{\mathbb{F}}_p$.)

Masses, or (weighted) counting on moduli spaces

The **mass** of a central leaf $\Lambda_{X_0} = \{X \in \mathcal{A}_g : X[p^\infty] \simeq X_0[p^\infty]\}$ is

$$\mathrm{Mass}(\Lambda_{X_0}) = \sum_{X \in \Lambda_{X_0}} \frac{1}{|\mathrm{Aut}(X)|}.$$

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For a generic **supersingular** abelian variety X_0 of dimension 3, we have

$$\mathrm{Mass}(\Lambda_{X_0}) = \frac{p^{3+2d}(p^2-1)(p^4-1)(p^6-1)}{2^{11} \cdot 3^4 \cdot 5 \cdot 7}.$$

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Consequence: every such X_0 has automorphism group $\mathrm{Aut}(X_0) = \{\pm 1\}$.

This was conjectured by Oort for any dimension g and characteristic p.
Oort's conjecture is **still open** for $g \geq 5$ odd. (We proved the rest :))

Thank you for your attention!