Reading seminar "Arithmetic of K3 surfaces"

- next week (27/1): 12.45 – 14.30?

Today: first definitions & properties of K3 surfaces, following Huybrechts Chapters 1 & 2

We will define algebraic, complex, and polarised K3 surfaces and study their cohomology & line bundles

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Let K be a field.

A variety over K is a separated, geometrically integral scheme of finite type over K, where

separated = "like Hausdorff"

integral = reduced + irreducible, every open $U = \mathrm{Spec}(R)$ for R an integral domain

geom. integral = s.t. $X_{K'}$ is integral $\forall\, K'/K$

finite type = s.t. under the structure map $X \to \mathrm{Spec}(K)$, every open $\mathrm{Spec}(R)$ is a finitely generated K-algebra.

$\Downarrow$

noetherian

An (algebraic) K3 surface is a complete, non-singular two-dimensional variety such that $\Omega^2_{X/k} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$, where

complete = "like compact"
non-singular = all local rings are regular ($\dim(R) = \dim(m/m^2)$)

$\Big\} \Rightarrow$ projective

$\Omega^2_X \simeq \mathcal{O}_X$:

- Cotangent sheaf Ω_X is locally free sheaf of $\mathcal{O}_X$-modules of rank 2 ($=$ tr.deg $K(X)/K$).
- $\Omega^2_X = \Lambda^2 \Omega_X$ (determinant)
  $= \omega_X$ (canonical sheaf, $\hookrightarrow K_X$ canonical line bundle)
- That is, alternating pairing $\Omega_X \times \Omega_X \to \omega_X \simeq \mathcal{O}_X$ is perfect.
- Tangent sheaf is $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ is locally free of rank 2 as well, so $\mathcal{T}_X \simeq \Omega_X$.

$H^1(X, \mathcal{O}_X) = 0$:

- "the irregularity of X is trivial"
- We know $H^i(X, \mathcal{O}_X) = 0 \ \forall\, i > 2$ since $\dim(X) = 2$
- $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$

Example 1 (smooth complete intersection)

For $i=1,2,\dots,r$, let $H_i : f(x_1,\dots,x_n)=0$ with $\deg(f)=d_i \geq 2$ in $\mathbb{P}^n_k$ be a hypersurface of degree d_i,
and let $X = H_1 \cap H_2 \cap \dots \cap H_r \subseteq \mathbb{P}^n_k$ of codimension r.

Then $\omega_{H_i} = \mathcal{O}_{H_i}(d_i - n - 1) \quad \forall i$

and $\omega_X = \mathcal{O}_X\left(\sum_{i=1}^{r} d_i - n - 1\right)$

$\omega_X \simeq \mathcal{O}_X$: we want $n-r=2$ and $\sum d_i - n - 1 = 0$.

(a) $n=3, r=1, d_1=4 \quad \Rightarrow$ quartic surface in $\mathbb{P}^3$

(b) $n=4, r=2, (d_1,d_2)=(2,3) \Rightarrow$ quadric $\cap$ cubic in $\mathbb{P}^4$

(c) $n=5, r=3, (d_1,d_2,d_3)=(2,2,2) \Rightarrow$ quadric $\cap$ quadric $\cap$ quadric in $\mathbb{P}^5$

$H^1(X,\mathcal{O}_X)=0$ (for (a)):

$\mathcal{O}_{\mathbb{P}^3}(-4) = \omega_{\mathbb{P}^3}$ and $\mathcal{O}_X = \omega_X$ fit into

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

Long exact sequence

$$\cdots \longrightarrow H^1(\mathbb{P}^3, \cancelto{0}{\mathcal{O}_{\mathbb{P}^3}}) \to H^1(\mathbb{P}^3, \mathcal{O}_X) \to H^2(\mathbb{P}^3, \cancelto{0}{\mathcal{O}_{\mathbb{P}^3}(-4)}) \to \cdots$$

but

$$\boxed{H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n_k}(d)) = 0 \quad \text{if } q \neq 0, n}$$

so $H^1(X,\mathcal{O}_X) \hookrightarrow H^1(\mathbb{P}^3, \mathcal{O}_X) = 0$.

Example 2 (Kummer surface)

Let $\mathrm{char}(K) \neq 2$ and let A/K be an abelian surface.

$\exists$ involution $\iota : A \to A$, $x \mapsto -x$ with fixed points $A[2]$.

We know that $\# A[2](\bar{K}) = 2^{2\dim(A)} = 2^4 = 16$.

Now blow up A in these fixed points: $\tilde{A} \to A$.

Then $\tilde{A}$ has involution $\tilde{\iota}$. Let $\boxed{X := \tilde{A}/\tilde{\iota}}$.

Then $\pi : \tilde{A} \to X$ is a double cover, ramified along exceptional divisors E_i $(i = 1, \dots, 16)$ with images $\bar{E}_i$, so

$$\pi^* \mathcal{O}(\bar{E}_i) = \mathcal{O}(2E_i). \tag{1}$$

Can prove that

$$\omega_{\tilde{A}} = \mathcal{O}_{\tilde{A}}(\Sigma E_i) \tag{2}$$

Hurwitz ramification formula gives

$$\omega_{\tilde{A}} \simeq \pi^* \omega_X \otimes \mathcal{O}_{\tilde{A}}(\underbrace{\Sigma E_i}_{\text{ramification divisor}}) \tag{3}$$

So (2) & (3) $\Rightarrow$

$$\pi^* \omega_X \simeq \mathcal{O}_{\tilde{A}} \tag{4}$$

$\underline{\omega_X \simeq \mathcal{O}_X}$:

<u>Projection formula:</u> Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, $\mathcal{E}$ a locally free $\mathcal{O}_Y$-module of finite rank. Then

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{E}) \simeq f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

So for us: $(\tilde{A}, \mathcal{O}_{\tilde{A}}) \to (X, \mathcal{O}_X)$ yields

$$\pi_* \mathcal{O}_{\tilde{A}} \otimes \omega_X \simeq \pi_*(\mathcal{O}_{\tilde{A}} \otimes \pi^* \omega_X) \overset{(4)}{\simeq} \pi_*(\mathcal{O}_{\tilde{A}} \otimes \mathcal{O}_{\tilde{A}})$$
$$\simeq \pi_*(\mathcal{O}_{\tilde{A}}) \tag{5}$$

$$\pi_* \mathcal{O}_{\tilde{A}} \otimes \omega_X \simeq \pi_* \mathcal{O}_{\tilde{A}} \qquad (5)$$

Also have $\pi_* \mathcal{O}_{\tilde{A}} \simeq \mathcal{O}_X \oplus L^{\otimes -1}$ for L s.t. $L^2 = \underline{\mathcal{O}_X}(\Sigma E_i)$ (6) $\updownarrow$ (1)

So $\mathcal{O}_X \oplus L^{-1} \underset{(5)\,\text{IS}}{\simeq} \pi_* \mathcal{O}_{\tilde{A}} \underset{(4)}{\simeq} \pi_* \pi^* \omega_X$

$$\pi_* \mathcal{O}_{\tilde{A}} \otimes \omega_X \underset{(6)}{\simeq} (\mathcal{O}_X \oplus L^{-1}) \otimes \omega_X \simeq \omega_X \oplus (L^{-1} \otimes \omega_X)$$

so either $\mathcal{O}_X \simeq L^{-1} \otimes \omega_X$ $\times$

or $\boxed{\mathcal{O}_X \simeq \omega_X}$

$\underline{H^1(X, \mathcal{O}_X) = 0}$: $H^1(X, \mathcal{O}_X) \hookrightarrow H^1(\tilde{A}, \mathcal{O}_{\tilde{A}}) = H^1(A, \mathcal{O}_A)$

is contained in subspace invariant under induced action of ι.

(these are the fixed points of ι on A) (?)

Line bundles & cohomology (Part I)

Let X be any smooth projective surface over a field K.
Let $\mathrm{Div}(X)$ be its group of (Weil = Cartier) divisors.

$\exists$ intersection pairing $(\ ,\ ) : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$
$$(C, D) \mapsto C.D$$
such that:
- $C.D = \#(C \cap D)$ if C & D are nonsingular curves meeting transversally
- the pairing is symmetric and additive
- the outcome depends on linear equivalence classes of C & D (defined below)

The Euler characteristic of a coherent sheaf is
$$\chi(X, \mathcal{F}) = \chi(\mathcal{F}) = \sum (-1)^i \dim H^i(X, \mathcal{F}) = \sum (-1)^i h^i(X, \mathcal{F})$$
Then $(L_1, L_2) = \chi(\mathcal{O}_X) - \chi(L_1^*) - \chi(L_2^*) + \chi(L_1^* \otimes L_2^*)$

Riemann-Roch: $\chi(L) = \dfrac{(L \cdot L \otimes \omega_X^*)}{2} + \chi(\mathcal{O}_X)$

$\exists$ equivalence relations on $\mathrm{Div}(X)$:

(1) Linear equivalence:
$C \sim_{lin} D$ if $C = D + \mathrm{div}(f)$, $f \in K(X)$

$\mathrm{Pic}(X) = \mathrm{Div}(X) / \sim_{lin}$

(2) Algebraic equivalence:
$C \sim_{alg} D$ if $\exists$ connected curve T, pts $0, 1 \in T$, divisor E in $X \times T$ flat over T, such that
$$E|_{X \times \{0\}} - E|_{X \times \{1\}} = C - D$$

$\mathrm{Pic}^0(X)$ = classes alg. equiv. to 0

$NS(X) = \mathrm{Pic}(X) / \mathrm{Pic}^0(X)$
Néron-Severi group

(3) Numerical equivalence:
$C \sim_{num} D$ if $(C, E) = (D, E)$ $\forall E \in \mathrm{Div}(X)$

$\mathrm{Pic}^\tau(X)$ = classes num. equiv. to 0

$\mathrm{Num}(X) = \mathrm{Pic}(X) / \mathrm{Pic}^\tau(X)$

$NS(X)$ and $\mathrm{Num}(X)$ are finitely generated; Picard number $\rho(X) = \mathrm{rk}\, NS(X)$.

Now let X be a K3 surface again.

Euler characteristic: $\chi(\mathcal{O}_X) = h^0(X,\mathcal{O}_X) - h^1(X,\mathcal{O}_X) + h^2(X,\mathcal{O}_X) = 1 - 0 + 1 = 2$

Arithmetic genus $p_a = h^2(X,\mathcal{O}_X) - h^1(X,\mathcal{O}_X) = 1$

Geometric genus $p_g = h^2(X,\mathcal{O}_X) = 1$

Riemann-Roch: $\chi(L) = \frac{L^2}{2} + 2$

$\pi_1^{et}(X) = \{1\}$ when $K = K^{sep}$: for $\tilde{X} \to X$ étale of degree d we get $\omega_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}$ and $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = d\,\chi(X,\mathcal{O}_X)$ but $h^1(\tilde{X},\mathcal{O}_{\tilde{X}}) = h^1(X,\mathcal{O}_X)$ so $d=1$.

$\Downarrow$

This implies that $\mathrm{Pic}(X)$ is torsion-free. (cf. direct proof in [H, Rem 2.5]):

Also, $\mathrm{Pic}(X) \simeq NS(X) \simeq \mathrm{Num}(X)$.

Hodge numbers $h^{p,q} = \dim H^q(X, \Omega_X^p) = h^q(X, \Omega_X^p)$:

$h^{0,0} = h^0(X,\mathcal{O}_X) = 1$

$h^{0,2} = h^0(X,\Omega^2) = h^0(X,\mathcal{O}_X) = 1$

$h^{2,0} = h^2(X,\mathcal{O}_X) = 1$

$h^{2,2} = h^2(X,\Omega^2) = h^2(X,\mathcal{O}) = 1$

$h^{0,1} = h^1(X,\mathcal{O}) = 0$

Can show $h^{2,1} = h^{1,2} = h^{1,0} = 0$

Can compute that $h^{1,1} = 20$.

↳ (chern classes, Noether's formula, other $h^{p,q}$)

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

↑ Hodge diamond

Complex K3 surfaces

A complex K3 surface is a compact connected two-dimensional complex manifold such that $\Omega^2_X = \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

For comparisons, we need:

GAGA: $\exists$ equivalence of abelian categories

separated scheme locally of finite type over $\mathbb{C}$	$X \longmapsto X^{an}$	complex space with set of points $X(\mathbb{C})$
(smooth)	$\longmapsto$ (manifold)	
coherent sheaf on X	$\mathcal{F} \longmapsto \mathcal{F}^{an}$	coherent sheaf on X^{an}
	$\mathcal{O}_X \longmapsto \mathcal{O}_{X^{an}}$	
	$\Omega_{X/\mathbb{C}} \longmapsto \Omega_{X^{an}}$	
	$H^*(X, \mathcal{F}) \xrightarrow{\sim} H^*(X^{an}, \mathcal{F}^{an})$	

Then $\{\text{algebraic K3 surfaces} / \mathbb{C}\} \hookrightarrow \{\text{complex K3 surfaces}\}$

$X \mapsto X^{an}$

image = projective complex K3's

This is useful because we can compute singular (co)homology on X^{an}:

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$
$$f \mapsto \exp(2\pi i f)$$

yields

$$0 \longrightarrow \mathbb{Z} \to \mathbb{C} \twoheadrightarrow \mathbb{C}^* \to 0$$
$$\longrightarrow 0 \to \boxed{\mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \longrightarrow \mathbb{C}}$$
$$\longrightarrow H^2(X, \mathcal{O}_X^*) \to 0 \to H^3(X, \mathcal{O}_X)$$

- Since $H^1(X, \mathbb{Z}) = 0 \Rightarrow H^3(X, \mathbb{Z}) = 0$ up to torsion (Poincaré duality) but can show $\nexists$ torsion.
- $H^2(X, \mathbb{Z})$ is (torsion) free abelian group.

So $\mathrm{Pic}(X) \to H^2(X,\mathbb{Z}) \to \mathbb{C}$ $\Rightarrow$ $H^2(X,\mathbb{Z})$ is torsion-free abelian group.

Further, $\mathrm{Pic}(X) \simeq H^2(X,\mathbb{Z}) \cap H^1(X,\Omega) \subseteq H^1(X,\Omega)$ and $h^{1,1} = 20$ so $\rho(X) \leq 20$. (Each rank is actually realised!)

N.B. over arbitrary fields we have $\rho(X) \leq 22$ since $H^2_{et}(X,\mathbb{Z}_\ell)$ has rank 22

Betti numbers $b_k := \sum_{p+q=k} h^{p,q}(X)$, so

$$\left.\begin{aligned} b_0 &= h^{0,0} = 1 \\ b_1 &= h^{1,0} + h^{0,1} = 0+0 = 0 \\ b_3 &= h^{1,2} + h^{2,1} = 0+0 = 0 \\ b_4 &= h^{2,2} = 1 \end{aligned}\right\}$$

Also $\sum (-1)^i b_i(X) = e(X) = c_2(X) = 24$

so $b_2 = 22 = h^{0,2} + h^{2,0} + h^{1,1}$
$= 1 + 1 + 20$

? so $\mathrm{rk}\, H^2(X,\mathbb{Z}) = 22$.

Topological intersection form = cup product

$$B : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \longrightarrow H^4(X,\mathbb{Z}) \cong \mathbb{Z}$$

is an even perfect bilinear pairing, giving rise to integral quadratic form

$$q : H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$x \longmapsto B(x,x),$$

where even = s.t. $B(x,x) \in 2\mathbb{Z}$ $\forall x$

perfect = s.t. $H^2(X,\mathbb{Z}) \simeq \mathrm{Hom}(H^2(X,\mathbb{Z}),\mathbb{Z})$

$\Downarrow$

$H^2(X,\mathbb{Z})$ is unimodular, i.e. Gram matrix of determinant ± 1.

Can show B has signature $(3,19)$, i.e.,

$q_{\mathbb{R}} : H^2(X,\mathbb{Z}) \otimes \mathbb{R} \to \mathbb{R}$ has 3 positive eigenvalues & 19 negative eigenvalues

Have hyperbolic plane $\mathcal{U}$ $\leftrightarrow$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ rank 2, eigenvalues ± 1

and $E_8(-1)$ $\leftrightarrow$ $\begin{pmatrix} -2 & & & & & & & \\ & -2 & & & & & & \\ & & -2 & & & & & \\ & & & -2 & & & & \\ & & & & -2 & & & \\ & & & & & -2 & & \\ & & & & & & -2 & \\ & & & & & & & -2 \end{pmatrix}$ rank 8, eigenvalues -2

Classification of unimodular lattices (cf. Serre "Cours d'Arithmetique") shows that

$$H^2(X, \mathbb{Z}) \simeq \Lambda_{K_3} := E_8(-1) \oplus E_8(-1) \oplus \mathcal{U} \oplus \mathcal{U} \oplus \mathcal{U}$$

Line bundles & Cohomology (Part II)

Let $k = \overline{k}$ be algebraically closed.

Let X be any smooth projective surface over k.

- A complete linear system on X is the set of all effective divisors linearly equivalent to a fixed divisor D_0, denoted $|D_0|$.
  Then $D_0 \leftrightarrow$ invertible sheaf L and $|D_0| \leftrightarrow (H^0(X,L) - \{0\}) / k^*$.
- A linear system ∂ is a subset of $|D_0|$, corresponding to a sub-vector space $V \subseteq H^0(X,L)$.
- A base point $P \in X$ of ∂ is such that $P \in \operatorname{supp}(D)$ $\forall D \in \partial$.
- The base locus $Bs|L|$ of $|L|$ is $\bigcap_{s \in H^0(X,L)} Z(s)$, maximal closed subscheme of X contained in all $D \in |L|$.

- $h^0(X,L) > 1$ induces $\varphi_L : X \dashrightarrow \mathbb{P}(H^0(X,L)^*)$, regular on $X \setminus Bs|L|$.

$\Rightarrow$ Linear systems without base points induce morphisms $X \to \mathbb{P}^n_k$.

A line bundel L is:

nef	if $(L.C) \geq 0$ for all closed curves $C \subseteq X$.
big and nef	if $L^2 > 0$ and L is nef
very ample	if φ_L is a closed immersion
ample	if $L^{\otimes r}$ is very ample for some r

Can decompose $\underset{\text{mobile}}{L = M} + \underset{\text{fixed}}{F}$

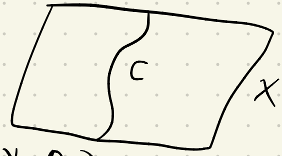
$$L = \underbrace{M}_{\text{mobile}} + \underbrace{F}_{\text{fixed}}$$

where F = one-dimensional part of $Bs|L|$

and $M = L(-F)$.

Then $H^0(X,M) \simeq H^0(X,L)$, M is nef and $M^2 \geq 0$.

Consider curve C in X.



It has arithmetic genus $p_a(C) = 1 - \underset{\chi(C,\mathcal{O}_C)}{\chi(\mathcal{O}_C)}$

$= 1 + \chi(X, \mathcal{O}(-C)) - \chi(X, \mathcal{O}_X)$

and geometric genus $p_g(C) = h^1(C, \mathcal{O}_C) = h^0(C, \omega_C)$

If C is non-singular then $p_a(C) = p_g(C) = g(C)$ "genus"

If C is singular then $p_a(C) = p_g(C) + h^0(\delta)$ with

$\delta = \nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C$ for $\nu: \tilde{C} \to C$ normalisation

$\boxed{\begin{array}{l} \underline{\text{Adjunction formula}}: \quad 2p_a(C) - 2 = (C \cdot \omega_X \otimes \mathcal{O}(C)) \\ (\text{since } \omega_C \simeq \omega_X \otimes L(C) \otimes \mathcal{O}_C \simeq (\omega_X \otimes \mathcal{O}(C))|_C.) \end{array}}$

N.B. For any curve C with $p_g(C) \geq 2$, ω_C^k is very ample for $k \geq 3$.

When C is smooth irreducible with $g \geq 2$,

ω_C is very ample $\Leftrightarrow$ C is not hyperelliptic.

$\boxed{\begin{array}{l} \text{Kodaira-Ramanujam vanishing} \\ \text{Let char}(K) = 0 \text{ and let } L \text{ be big and nef.} \\ \text{Then} \quad H^i(X, L \otimes \omega_X) = 0 \quad \forall i > 0. \end{array}}$

Now let X be a K3 surface again.

Let $C \subset X$ be a curve.

- Adjunction formula: $2p_a(C) - 2 = (C \cdot \omega_X \otimes \mathcal{O}(C)) = C^2$

  (since $\omega_C \simeq \mathcal{O}(C)|_C$.)

  So $C^2 \geq -2$.

  A (-2)-curve C has $C^2 = -2$. Then $p_a(C) = 0 = p_g(C)$, so C is smooth and hence $(K = \bar{K})$ $C \simeq \mathbb{P}^1$.

- For C smooth and $L = \mathcal{O}(C)$, we have

$$0 \to \underset{1}{H^0(X, \mathcal{O})} \to \underset{g+1}{H^0(X, L)} \twoheadrightarrow H^0(C, L|_C) \to 0$$

$$\overset{g}{H^0(X, L|_C)} = H^0(C, L|_C) \text{ is } H^0(C, \omega_C)$$

So $h^0(X, L) = g+1 = \chi(X, L)$ $(= \frac{L^2}{2} + 2)$ so $H^1(X, L) = 0$

In fact $H^0(X, L^\ell) \twoheadrightarrow H^0(C, L^\ell|_C)$ $\forall \ell > 0$.

- Let C be a smooth irreducible curve in X of genus $g \geq 1$. Then $L = \mathcal{O}(C)$ is base-point free and $\varphi_L : X \to \mathbb{P}^g$ restricts to $C \to \mathbb{P}^{g-1}$.

<u>Proof</u>: $\varphi_{\omega_C} : X \to \mathbb{P}(H^0(X, \omega_C)^*)$

But by the above,

$$\mathbb{P}(H^0(X, \omega_C)^*) = \mathbb{P}(H^0(X, L|_C)^*) \subseteq \mathbb{P}(H^0(X, L)^*) = \mathbb{P}^g$$
$$= \mathbb{P}(H^0(C, \omega_C)^*)$$

Base points are only at C but $L|_C = \omega_C$ has no base points even there. $\square$

<u>N.B.</u> For $g = 2$ we get hyperelliptic curve C, so $C \to \mathbb{P}^1$ (of degree 2) is the restriction of $X \to \mathbb{P}^2$ (of degree 2, branched along a sextic). $(\mathrm{char}(K) \neq 2)$

- We can decompose any line bundle L on X as $L = M + F$, where $F = \sum a_i C_i$ with $a_i \geq 0$ and $C_i \simeq \mathbb{P}^1$ and M is semi-ample and $M^2 \geq 0$

$\qquad\qquad\qquad\qquad$ $\overset{\shortparallel}{\text{s.t. } M^{\otimes r}}$ is base point-free for some r.

e.g. if $|L|$ is a complete linear system, then $Bs\,|L| = F$.

$\qquad$ (This is clear when $L = \mathcal{O}(F)$; then $M = L(-F) = \mathcal{O}$ is base-point free.)

- If L is such that $L^2 > 0$ and $|L|$ contains an irreducible curve, then $|L|$ is base point-free.

(If $\mathrm{char}(K) \neq 2$ and C is irreducible and $C^2 > 0$, then the generic curve in $|C|$ is smooth and irreducible.)

Kodaira-Ramanujam vanishing $\quad H^i(X, L) = 0 \quad \forall i > 0$ $\quad$ when L is big & nef

This holds even when $\mathrm{char}(K) \neq 0$!

We saw $H^1(X, L) = 0$ for $L = \mathcal{O}(C)$ when C smooth.

Direct proof (for $H^1(X, L) = 0$ when $L = \mathcal{O}(C)$ for C connected & reduced)

$$0 \to \mathcal{O}(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0 \qquad \text{induces}$$

$$0 \to \underset{0}{\cancel{H^0(X, \mathcal{O}(-C))}} \to H^0(X, \mathcal{O}_X) \xrightarrow[\sim]{res} H^0(X, \mathcal{O}_C) \to \underset{H^1(X,L)^*}{\underset{\shortparallel}{H^1(X, \mathcal{O}(-C))}} \hookrightarrow \underset{0\ (K3)}{\cancel{H^1(X, \mathcal{O}_X)}}$$

We have $H^0(X, \mathcal{O}(-C)) = H^0(X, L^*) = 0$

and $H^1(X, \mathcal{O}(-C)) = H^1(X, L^*) \simeq H^1(X, L \otimes \omega_X)^*$

and $H^0(X, \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_C) \simeq H^0(C, \mathcal{O}_C)$ (?)

so $H^1(X, L)^* \hookrightarrow 0$ hence it is trivial.

Polarised K3 surfaces

A $\left\{\begin{array}{l}\text{polarised}\\ \text{quasi-polarised}\end{array}\right\}$ K3 surface of degree d is a K3 surface X with an

$\left\{\begin{array}{l}\text{ample}\\ \text{big and nef}\end{array}\right\}$ line bundle L, $\boxed{\text{write } (X, L)}$

such that $L^2 = 2d$ and L is primitive
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \parallel$
indivisible in $\mathrm{Pic}(X)$.

A smooth curve in $|L|$ then has genus g such that $2g-2 = 2d$
(Adjunction: $2g-2 = C^2$.)

Then $X \to \mathbb{P}^g$ and we say that X has "genus g."

$\boxed{\text{Fact}}$ When $k = \bar{k}$ there exists such a surface for any $g \geq 3$.

↳ This is not true e.g. over finite fields, where the degree is bounded!