K3 seminar talk 13

(Twisted) Derived equivalences & rational points)

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Derived equivalences: finite field case

Recall from Huybrechts Ch4 appendix / Stefano's talk:

Let X be a K3 over $\mathbb{F}_q$, let $\bar{X} = X \times \bar{\mathbb{F}}_q$, in char. p.

Have a relative Frobenius morphism $f : \bar{X} \to \bar{X}$ such that, $\forall r \geq 1$,

$$\boxed{X(\mathbb{F}_{q^r}) = \{\text{points of } \bar{X} \text{ fixed by } f^r\} = \sum_i (-1)^i \, tr\left(f^{r*} \,|\, H^i_{et}(\bar{X}, \mathbb{Q}_\ell)\right)}$$

fitting into a generating function $Z(X,t) = \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})| \frac{t^r}{r}\right)$.

Weil conjectures $\quad Z(X,t)^{-1} = (1-t)\left(\prod_{i=1}^{22}(1-\alpha_i t)\right)(1-q^2 t) \in \mathbb{Z}(t)$

where $\{\alpha_1, \ldots, \alpha_{22}\} = \{\frac{q^2}{\alpha_1}, \ldots, \frac{q^2}{\alpha_{22}}\}$ since $\alpha_{2j-1} \cdot \alpha_{2j} = q^2$,

$|\alpha_i| = q \;\forall i$ and $\alpha_i = \pm q$ for $i = 1, \ldots, 2k$

Theorem (Lieblich-Olsson) Let X, Y be K3 surfaces over $\mathbb{F}_q$.
If $D^b(X) \simeq D^b(Y)$, then $Z(X,t) = Z(Y,t)$.

Proof Write $\Phi : D^b(X) \xrightarrow{\sim} D^b(Y)$. By Orlov's theorem (Dirk's talk),

$\Phi = \Phi_P$ is a Fourier-Mukai transform for some $P \in D^b(X \times Y)$ / $\mathbb{F}_q$.

(cf. derived Torelli: $v(P)$ is integral, so get $\varphi_P^H : H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ Hodge isometry)

$v(P)$ is f-invariant, so get f-equivariant isomorphism

$$H^0_{et}(\bar{X}, \mathbb{Q}_\ell) \oplus H^2_{et}(\bar{X}, \mathbb{Q}_\ell(1)) \oplus H^4_{et}(\bar{X}, \mathbb{Q}_\ell(2)) \simeq H^0_{et}(\bar{Y}, \mathbb{Q}_\ell) \oplus H^2_{et}(\bar{Y}, \mathbb{Q}_\ell(1)) \oplus H^4_{et}(\bar{Y}, \mathbb{Q}_\ell(2))$$

So sets of eigenvalues of f must coincide:

$$\{\alpha_0\} \cup \{\alpha_1, \ldots, \alpha_{22}\} \cup \{\alpha_{4,1}\} = \{\beta_0\} \cup \{\beta_1, \ldots, \beta_{22}\} \cup \{\beta_{4,1}\}$$

Comparing absolute values finishes the proof. $\square$

Quick intro to Brauer groups

Let K be a field.

Then the Brauer group of K is

$$Br(K) := \{\text{central simple algebras} / K\} / \sim$$

where $A \sim A' \iff A \otimes M_n(K) \simeq A' \otimes M_m(K)$

$$\simeq H^2_{ét}(G_K, K^{sep \times})$$

Let X be a regular integral scheme over a field K.

An Azumaya algebra $\mathcal{A}$ over X is an $\mathcal{O}_X$-algebra, coherent as $\mathcal{O}_X$-module, étale locally $\simeq M_n(\mathcal{O}_X)$, such that $\mathcal{A}(x) = \mathcal{A} \otimes \kappa(x)$ is a CSA$/\kappa(x)$, $\forall x \in X$.

Then the Brauer group of X is

$$Br(X) = \{\text{Azumaya algebras over } X\} / \sim$$

where $\mathcal{A} \sim \mathcal{A}' \iff \mathcal{A} \otimes End(E_1) \simeq \mathcal{A}' \otimes End(E_2)$,

for E_1, E_2 locally free sheaves

$$\simeq H^1_{ét}(X, PGl_n)$$

$$\simeq H^2_{ét}(X, \mathbb{G}_m)_{tor} = H^2(X, \mathcal{O}_X^*)_{tor}$$

It is a finite abelian torsion group with operation $\otimes$.

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A twisted K3 surface is a pair (X, α), where X is a K3-surface and $\alpha \in Br(X)$.

$Coh(X, \alpha)$ = category of twisted sheaves on X:

choosing a representative $\{\alpha_{ijk} \in \mathcal{O}^*(U_{ijk})\}$, such a sheaf is $(\{E_i\}, \{\varphi_{ij}\})$ where E_i is a coherent sheaf on U_i and $\varphi_{ij} : E_j|_{U_i \cap U_j} \xrightarrow{\sim} E_i|_{U_i \cap U_j}$ satisfy

$\varphi_{ii} = id$, $\varphi_{ji} = \varphi_{ij}^{-1}$, and $\boxed{\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot id}$

Why would we study twisted K3 surfaces?

$$\boxed{\text{A **twisted K3 surface** is a pair } (X, \alpha), \text{ where } X \text{ is a K3-surface and } \alpha \in Br(X).}$$

We saw in Dirk's talk: if $M_H(v)^s$ is a fine moduli space,
then $D^b(X) \simeq D^b(M_H(v)^s)$.

And in Martijn's talk: fine moduli space $\Leftrightarrow$ $\exists$ universal family $\mathcal{E}$ on $M_H(v)^s \times X$.

Idea: a twisted universal family always exists!

Start with candidate sheaf $\mathcal{E}'$ semistable (constructed like in M's talk)

and open cover $\bigcup_i U_i$ of M. Denote $\mathcal{E}'_i = \mathcal{E}'|_{U_i \times X}$.

Then $\mathcal{E}_j|_{(U_i \times U_j) \times X} \simeq \mathcal{E}_i|_{(U_i \times U_j) \times X} \otimes p^* \mathcal{L}_{ij}$ for $\mathcal{L}_{ij} = p_* Hom(\mathcal{E}'_i, \mathcal{E}'_j)$

and $\exists\, \xi_{ij} : \mathcal{L}_{ij} \xrightarrow{\sim} \mathcal{O}_{U_{ij}}$ such that

$$\alpha_{ijk} := (\xi_{ij} \otimes \xi_{jk}) \circ \xi_{ik}^{-1} \in \Gamma(U_{ijk}, \mathcal{O}^*)$$

$$\Rightarrow \alpha \in Br(M_H(v)^s)$$

really $0 \to \langle \alpha \rangle \to Br(M_H(v)^s) \to Br(X) \to 0$

So we have an $\alpha \boxtimes 1$ - twisted universal family on $M_H(v)^s \times X$

$$\Rightarrow \boxed{D^b(X) \simeq D^b(M_H(v)^s, \alpha^{-1})}$$

"twisted F-M partners"

Let X be a complex K3,

$\rightarrow \mathrm{Coh}(X) \rightarrow D^b(X)$.

- For $E, F \in D^b(X)$ we have

  $$\chi(E,F) = -\underbrace{\langle v(E), v(F) \rangle}_{\text{Mukai pairing}}$$

  for the Mukai vector

  $$v = ch \cdot \sqrt{td(X)}$$

- Put a weight 2 Hodge structure on

  $H^*(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) \overset{\wedge\,\oplus U}{\oplus} U, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

  $\widetilde{H}(X,\mathbb{Z}) = \widetilde{H}^{2,0}(X) \oplus \widetilde{H}^{1,1}(X) \oplus \widetilde{H}^{0,2}(X)$

  where

  $\widetilde{H}^{2,0}(X) := H^{2,0}(X,\mathbb{Z}) = \langle \sigma \rangle$

  $\widetilde{H}^{1,1}(X) := H^{1,1}(X,\mathbb{Z}) \oplus U$

  so that

  $\widetilde{H}^{2,0}(X) \perp \widetilde{H}^{1,1}(X)$ w.r.t. Mukai pairing

Let (X,α) be a twisted K3.

$\rightarrow \mathrm{Coh}(X,\alpha) \rightarrow D^b(X,\alpha)$.

- For $E, F \in D^b(X,\alpha)$ we have

  $$\chi(E,F) = -\langle v^B(E), v^B(F) \rangle$$

  for the twisted Mukai vector

  $$v^B = ch^B \cdot \sqrt{td(X)}$$

Put a weight 2 Hodge structure on

$H^*(X,\mathbb{Z})$:

$\widetilde{H}(X, \alpha_B, \mathbb{Z})$

where

$\widetilde{H}^{2,0}(X,\alpha_B) = \exp(B) \cdot \widetilde{H}^{2,0}(X) = \langle \sigma + B \wedge \sigma \rangle$

$\widetilde{H}^{1,1}(X,\alpha_B) = \exp(B) \cdot \widetilde{H}^{1,1}(X)$

so that

$\widetilde{H}^{2,0}(X,\alpha_B) \perp \widetilde{H}^{1,1}(X,\alpha_B)$

wrt Mukai pairing

Recall exponential sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0 \implies$

$$\cdots \rightarrow \underset{0}{H^1(X,\mathcal{O}_X)} \rightarrow \underset{\mathrm{Pic}(X)}{\underset{\parallel}{H^1(X,\mathcal{O}_X^*)}} \rightarrow H^2(X,\mathbb{Z}) \rightarrow H^2(X,\mathcal{O}_X) \xrightarrow{\exp} H^2(X,\mathcal{O}_X^*) \rightarrow \underset{0}{H^3(X,\mathbb{Z})}$$

so for our fixed $\alpha \in H^2(X,\mathcal{O}_X^*) = H^2(X,\mathcal{O}_X)/H^2(X,\mathbb{Z})$

we find $B \in H^2(X,\mathcal{O}_X) = H^{0,2} \subset H^2(X,\mathbb{R}) \implies$ write $\alpha = \alpha_B$.

(& since α is torsion, $B \in H^2(X,\mathbb{Q})$), unique up to $H^2(X,\mathbb{Z})$ and $\mathrm{Pic}(X)$.

Let $\exp(B) = 1 + B + \frac{B^2}{2} \in H^*(X,\mathbb{Q})$; this preserves the Mukai pairing

N.B.: • $\exp(B) : \widetilde{H}(X,\mathbb{Z}) \rightarrow \widetilde{H}(X,\alpha,\mathbb{Z})$ is a Hodge isometry

Let X be a complex K3.
$\to \mathrm{Coh}(X) \to D^b(X)$.

- For $E, F \in D^b(X)$ we have
$$\chi(E,F) = -\langle v(E), v(F)\rangle$$
for the Mukai vector Mukai pairing
$$v = ch \cdot \sqrt{td(X)}$$

- Put a weight 2 Hodge structure on
$H^*(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) \oplus U$, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: ^ ou
$$\widetilde{H}(X,\mathbb{Z}) = \widetilde{H}^{2,0}(X) \oplus \widetilde{H}^{1,1}(X) \oplus \widetilde{H}^{0,2}(X)$$
where
$$\widetilde{H}^{2,0}(X) := H^{2,0}(X,\mathbb{Z}) = \langle \sigma \rangle$$
$$\widetilde{H}^{1,1}(X) := \widetilde{H}^{1,1}(X,\mathbb{Z}) \oplus U$$
so that
$\widetilde{H}^{2,0}(X) \perp \widetilde{H}^{1,1}(X)$ w.r.t. Mukai pairing

Let (X,α) be a twisted K3.
$\to \mathrm{Coh}(X,\alpha) \to D^b(X,\alpha)$.

- For $E, F \in D^b(X,\alpha)$ we have
$$\chi(E,F) = -\langle v^B(E), v^B(F)\rangle$$
for the twisted Mukai vector
$$v^B = ch^B \cdot \sqrt{td(X)}$$

Put a weight 2 Hodge structure on
$H^*(X,\mathbb{Z})$:
$$\widetilde{H}(X,\alpha_B,\mathbb{Z})$$
where
$$\widetilde{H}^{2,0}(X,\alpha_B) = \exp(B)\cdot\widetilde{H}^{2,0}(X) = \langle \sigma + B\wedge\sigma \rangle$$
$$\widetilde{H}^{1,1}(X,\alpha_B) = \exp(B)\cdot\widetilde{H}^{1,1}(X)$$
so that
$\widetilde{H}^{2,0}(X,\alpha_B) \perp \widetilde{H}^{1,1}(X,\alpha_B)$
wrt Mukai pairing

Mukai pairing $\langle \alpha,\beta\rangle = (\alpha_2.\beta_2) - (\alpha_0.\beta_4) - (\alpha_4.\beta_0)$ has signature $(4,20)$
$\Rightarrow$ for ample class $\ell \in H^{1,1}(X,\mathbb{Z})$, have 4-dim. positive-definite $H_X(\ell) \subseteq \widetilde{H}(X,\mathbb{R})$,
spanned by $\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), \mathrm{Re}(\exp(i\ell)), \mathrm{Im}(\exp(i\ell))$
"four positive directions, in their natural orientation"
where $\exp(i\ell) = (1, i\ell, -\ell^2/2)$. Choose $\ell = \omega$ Kähler class!

Derived Torelli theorem:
$D^b(X) \simeq D^b(Y) \iff$
$\widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\mathbb{Z})$ Hodge isometry

Twisted derived Torelli theorem:
$D^b(X,\alpha) \simeq D^b(Y,\beta) \iff$
$\widetilde{H}(X,\alpha,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\beta,\mathbb{Z})$ Hodge isometry
respecting natural orientation of 4 positive dimensions

Derived Torelli theorem:	Twisted derived Torelli theorem:
$D^b(X) \simeq D^b(Y) \iff \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\mathbb{Z})$ Hodge isometry	$D^b(X,\alpha) \simeq D^b(Y,\beta) \iff \widetilde{H}(X,\alpha,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\beta,\mathbb{Z})$ Hodge isometry respecting natural orientation of 4 positive dimensions

Remarks:

1) We saw in Sergej's talk that $\Phi_{\mathcal{P}}: D^b(X) \xrightarrow{\sim} D^b(Y)$ gives Hodge isometry $\varphi^H_{\mathcal{P}}: H^*(X,\mathbb{Q}) \xrightarrow{\sim} H^*(Y,\mathbb{Q})$ $\left(\Rightarrow \varphi^H_{\mathcal{P}}: \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\mathbb{Z}) \right)$

Can prove a refined statement:

[For Hodge isometry $\varphi: \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\mathbb{Z})$,
$\varphi = \varphi^H_{\mathcal{P}}$ coming from some Fourier-Mukai equivalence $\Phi_{\mathcal{P}}: D^b(X) \to D^b(Y)$
$\iff$ φ respects natural orientation of 4 positive dimensions.

So orientation-preservation is automatically satisfied.

In fact, Huybrechts-Macrì-Stellari prove that $\varphi^H_{\mathcal{P}} \neq (-\mathrm{id}_{H^2}) \oplus \mathrm{id}_{H^0 \oplus H^4}$

2) A Fourier-Mukai transform $\Phi_{\mathcal{P}}: D^b(X,\alpha) \simeq D^b(Y,\beta)$ has kernel

$$\mathcal{P} \in D^b(X \times Y, \alpha^{-1} \boxtimes \beta)$$

and gives Hodge isometry $\varphi^{\alpha,\beta}: H^*(X,\mathbb{Q}) \xrightarrow{\sim} H^*(Y,\mathbb{Q})$,

yielding isomorphisms $\bigoplus_{p-q=i} H^{p,q}(X,\alpha,\mathbb{Q}) \xrightarrow{\sim} \bigoplus_{p-q=i} H^{p,q}(Y,\beta,\mathbb{Q}) \quad \forall i.$

3) It follows from the derived Torelli theorem that

$D^b(X) \simeq D^b(Y) \iff T(X) \xrightarrow{\sim} T(Y)$ Hodge isometry of transcendental lattices
($T(X) \perp NS(X)$ in $H^2(X,\mathbb{Z})$.)

The analogue $D^b(X,\alpha) \simeq D^b(Y,\beta) \iff T(X,\alpha) \simeq T(Y,\beta)$ does not hold:

- $T(X,\alpha) \simeq T(Y,\beta) \not\Rightarrow$ Hodge isometry $\widetilde{H}(X,\alpha,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\beta,\mathbb{Z})$
- Hodge isometry $\widetilde{H}(X,\alpha,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\beta,\mathbb{Z})$ might not preserve orientation of positive directions

Consequences of derived equivalences

- Huybrechts-Stellari ① derived equivalent twisted K3 surfaces have isomorphic periods.
- Hassett-Tschinkel ② derived equivalent K3 surfaces X, Y over K of char $\neq 2$ have $\mathrm{Pic}(X) \simeq \mathrm{Pic}(Y)$ and $\mathrm{Br}(X)[n] \simeq \mathrm{Br}(Y)[n]$ if $(n, \mathrm{char}\, K) = 1$.

  ③ derived equivalent K3's over any field K have the same index:

  $\mathrm{ind}(X) = \{\gcd$ of degrees of K'/K such that $X(K') \neq \emptyset\}$

  (② ⇒) ④ If $D^b(X) \simeq D^b(Y)$ and X is elliptic, then Y is also elliptic.

$\boxed{\text{Question}}$ If $D^b(X) \simeq D^b(Y)$ and $X(K) \neq \emptyset$, do we have $Y(K) \neq \emptyset$?

N.B. When $K = \mathbb{F}_q$, we have seen that $D^b(X) \simeq D^b(Y) \Rightarrow Z(X,t) = Z(Y,t)$

$\Rightarrow |X(K')| = |Y(K')| \quad \forall K'/K$.

When $K = \mathbb{R}$, the answer is also Yes

a) Real varieties have a rational point iff their index is 1, so use ③

b) Equivalence / Mukai lattice captures topological type & hence manifold $X(\mathbb{R})$, so $X(\mathbb{R})$ and $Y(\mathbb{R})$ are diffeomorphic.

When $K = \mathbb{C}((t)) = \mathrm{Frac}(R)$ for $R = \mathbb{C}[[t]]$,

X/K has a model $\mathcal{X} \to \mathrm{Spec}(R)$ (with generic fibre X), so $X(K) \neq \emptyset \Leftrightarrow$ $\mathcal{X} \to \mathrm{Spec}(R)$ admits a section

Can show $X(K) \neq \emptyset$ if

A) X has a model with at most rational double points in central fibre $\mathcal{X}_0$, or

B) the quasi-unipotent monodromy action $T: H^2(\overline{X}, \mathbb{Z}) \to H^2(\overline{X}, \mathbb{Z})$ has trace $\neq -2$

Furthermore:

A) $\Rightarrow$ Having such an ("ADE") model is a derived invariant.

So if $D^b(X) \simeq D^b(Y)$ and $\exists$ ADE models, then $X(K), Y(K) \neq \emptyset$.

B) $\Rightarrow$ $D^b(X) \simeq D^b(Y) \Rightarrow$ Mukai lattices are monodromy-equivariantly isomorphic.

So if $D^b(X) \simeq D^b(Y)$ and traces are $\neq 2$, then $X(K), Y(K) \neq \emptyset$.

When $K =$ p-adic field, its residue field is some $\mathbb{F}_q$.

So via Hensel's lemma we can show:

$D^b(X) \simeq D^b(Y)$ and X & Y have good reduction $\Rightarrow X(K) \neq \emptyset$ iff $Y(K) \neq \emptyset$.

(Applying resolutions and deformations, this can be extended to X, Y admitting regular models satisfying A), when $p \geq 7$.)

Twisted question If X, Y are K3's over K and $D^b(X,\alpha) \simeq D^b(Y,\beta)$, do we have $(X,\alpha)(K) \neq \emptyset \Rightarrow (Y,\beta)(K) \neq \emptyset$?

Here, $x \in (X,\alpha)(K)$ means $x \in X(K)$ s.t. $\alpha(x) = 0 \in Br(K)$.

Ascher-Dasaratha-Perry-Zhou: "No" when K is $\mathbb{Q}, \mathbb{Q}_2$, or $\mathbb{R}$.

sketch of argument:

$$\begin{tikzcd}
 & Y \arrow[d] \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\
X_1 \arrow[d] & \mathbb{P}^2 \times \mathbb{P}^2 \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & X_2 \arrow[d] \\
\mathbb{P}^2 & & \mathbb{P}^2
\end{tikzcd}$$

s.t. π_i is surface fibration branched over sextic curve

$\Rightarrow$ the double cover of $\mathbb{P}^2$ branched at this sextic is a K3, X_i.

Moreover, X_i comes equipped with a Brauer class α_i, and $D^b(X_1,\alpha_1) \simeq D^b(X_2,\alpha_2)$. (unless char $K = 2$).

Can choose ramification divisor for $Y \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ so that

1) α_1 yields Brauer-Manin obstruction, so $X_1(\mathbb{Q}) = \emptyset$;
2) explicit computations yield $x \in X_2(\mathbb{Q})$;
3) more explicit computations and proof of 1) give cases $\mathbb{Q}_2, \mathbb{R}$.