K3 2021 02 23

6.5: Kulikov models: (special) degenerations of K3 surfaces over a base of dim. 1.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \\ \Delta \subset \mathbb{C} \end{tikzcd}$$

X smooth threefold, Δ disk

π proper, flat, surjective

$\forall t \neq 0$: X_t K3

X_0 possibly non-reduced, reducible, arbitrarily singular.

Modification:
$$\begin{tikzcd} X' \arrow[rr] \arrow[dr] & \circlearrowright & X \arrow[dl] \\ & \Delta & \end{tikzcd}$$
isom. over $\Delta^* = \Delta - \{0\}$.

Semistable degeneration: X_0 reduced with local normal crossings.

[KKMS-D]: After base change $\Delta \to \Delta$, $z \mapsto z^m$, every degeneration $X \to \Delta$ admits a modification that is semistable.

Assume $X \to \Delta$ is semistable. Kulikov, Persson–Pinkham: if the irred. cpts. Y_i of X_0 are Kähler, then $\exists$ a modification

$X' \to \Delta$ with trivial canonical bundle $\omega_{X'}$.

Such $X' \to \Delta$ is a "Kulikov model" of the original degeneration.

<u>Thm.</u> Let $X \to \Delta$ be a Kulikov degeneration. Then

I: X_0 is a smooth K3 surface, or

II: X_0 is a chain of elliptic ruled surfaces Y_i with rational surfaces on either end and s.th. for $i \neq j$

$D_{ij} := Y_i \cap Y_j$ is ($\emptyset$ or) elliptic or "nodal elliptic", or

III: X_0 is a union of rational surfaces Y_i s.th. the dual graph of X_0 is a triangulation of S^2, and for i fixed the D_{ij} with $j \neq i$ form a cycle of (at most nodal) rational curves on Y_i.

The Kulikov model is in general not unique.

$X \to \Delta$ yields a monodromy representation:

$$\rho : \mathbb{Z} \simeq \pi_1(\Delta^*, t) \longrightarrow O(H^2(X_t, \mathbb{Z})) \qquad T := \rho(1).$$

Known: T is quasi-unipotent: $\exists\, m, n \in \mathbb{Z}_{\geq 1}$: $(T^m - \mathrm{id})^n = 0$.

Choose m, n minimal. Semistable $\Rightarrow$ $m = 1$ (T unipotent).

$z \mapsto z^m$ sends T to T^m, so after base change the monodromy becomes unipotent.

<u>Thm.</u> A Kulikov model exists $\Leftrightarrow$ the monodromy is unipotent.

For a Kulikov model (so $m=1$):

type I if $n=1$

type II if $n=2$

type III if $n=3$.

Remarks:

1) Olsson: partial compactifications of the moduli space of polarized K3 surfaces using log smooth K3 surfaces.

2) Liedtke-Matsumoto: arithmetic analogue of the thm. just stated, detecting whether a K3 surface has good reduction.

6.4 Moduli spaces of polarized K3 surfaces via periods and applications

$/\mathbb{C}$ Jack, Ch. 5: moduli functor $\mathcal{M}_d$, coarse mod. sp. M_d

Stefano, 1st half Ch. 6: $\tilde{O}(\Lambda_d)\backslash D_d$, q.-proj. var.

Idea: construct M_d as an open subv. of this ↑ variety.

Recall: $D \subset \mathbb{P}(\Lambda_{\mathbb{C}})$ the period domain (Λ K3 lattice rk. 22):

D is open (classically) in a quadric in $\mathbb{P}(\Lambda_{\mathbb{C}})$ (dim. 20)

$\ell := e_1 + d f_1$ (in first copy of U), $\Lambda_d = \ell^{\perp}$, $D_d \subset \mathbb{P}(\Lambda_{d\,\mathbb{C}})$, dim. 19,

$D_d = D \cap \mathbb{P}(\Lambda_{d\,\mathbb{C}})$.

$$\tilde{O}(\Lambda_d) = \{ g|_{\Lambda_d} \mid g \in O(\Lambda),\ g(e_1 + d f_1) = e_1 + d f_1 \}$$

= subgroup of $O(\Lambda_d)$ that acts trivially on $(\Lambda_d)^* / \Lambda_d$.

$P(t) = dt^2 + 2$, $N := P(3) - 1$, $\mathrm{Hilb} := \mathrm{Hilb}^{P(3t)}_{\mathbb{P}^N}$, $\exists\, H \subset \mathrm{Hilb}$, open, that parametrizes polarized K3's (X, L) with $X \hookrightarrow \mathbb{P}^N$ and $L^3 \simeq \mathcal{O}(1)|_X$. We want a categorical quotient of H by

$PGL := PGL(N+1)$

$\mathrm{Pol} := \mathrm{Pol}(N+1)$.

Just as last time, from H we obtain an étale cover $\tilde{H}$, cx. mfd., principal $\tilde{O}(\Lambda_d)$-bundle assoc. with $\ell^\perp \subset R^2 f_* \mathbb{Z}$ (where $f: X \to H$ is the universal family).

($\ell = c_1(L) \in \Gamma(H, R^2 f_* \mathbb{Z})$; $\tilde{H}$ parametrizes isometries $\ell_t^\perp \simeq \Lambda_d$ that extend to $H^2(X_t, \mathbb{Z}) \simeq \Lambda$ (and send ℓ to $e_1 + d f_1$)).

On the pull-back family $\tilde{f}: \tilde{X} \to \tilde{H}$ we have a marking
$$R^2 \tilde{f}_* \mathbb{Z} \simeq \underline{\Lambda}, \quad \ell \mapsto e_1 + d f_1 .$$

The period map $\mathcal{P}_d : \tilde{H} \to D_d$ descends to
$$\overline{\mathcal{P}}_d : H \to {}_{\tilde{O}(\Lambda_d)}\backslash D_d .$$

Recall also: N_d, m.sp. of primitively polarized marked K3's of degree $2d$ (triples (X, L, φ), L ample, $\varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ isometry s.th. $\varphi(L) = e_1 + d f_1$).

N_d is a cx. mfd which is Hausdorff. (N is <u>not</u> Hausdorff.)

Now: $H/_{PGL} = \tilde{O}(\Lambda_d) \backslash N_d$: sets of orbits.

Global Torelli: $\bar{\mathcal{P}}_d : \tilde{O}(\Lambda_d) \backslash N_d \longrightarrow \tilde{O}(\Lambda_d) \backslash D_d$ is injective. (And $\mathcal{P}_d : N_d \to D_d$ also.)

Alternative formulation:

$(X, L) \cong (X', L') \iff \exists$ Hodge isometry $H^2(X, \mathbb{Z}) \simeq H^2(X', \mathbb{Z})$ mapping ℓ to ℓ'.

So $H/_{PGL}$ injects into $\tilde{O}(\Lambda_d) \backslash D_d$.

By Local Torelli, H/PGL is an open subset (classical topology).

Thm. of Borel (applied here): Y nonsingular cx. variety, $\varphi: Y \to \Gamma\backslash D_d$ a holom. map, where $\Gamma \subset \tilde{O}(\Lambda_d)$ is a finite index torsion-free subgroup (so $\Gamma\backslash D_d$ is smooth), then φ is algebraic. (!)

Replace $H \to \tilde{O}(\Lambda_d)\backslash D_d$ by $\Gamma\backslash\tilde{H} \to \Gamma\backslash D_d$.
Then the second map is algebraic, and it follows the first one is.

Cor. H/PGL is a q.-proj. var. M_d, which is a coarse m. sp. for $\mathcal{M}_d$, functor of primitively polarized K3 surfaces of degree d. $(/\mathbb{C})$.

M_d is Zariski open in $\overline{M}_d = \widetilde{O}(\Lambda_d) \backslash D_d$ (q.-proj.).

On the proof: $H \to \overline{M}_d$ is algebraic, so the image is constructible; but it is also analytically open, so Zariski open.

Still need to construct $\mathcal{M}_d \to \underline{M_d}$: rather similar to the above.

Cor. M_d is irreducible. (Since $\tilde{O}(\Lambda_d)$ interchanges D_d^+, D_d^-.)

$\overline{M}_d$ is best viewed as a m. sp. of polarized 'singular K3's' (surfaces with RDP's whose minimal resolutions are K3 surfaces). ($\overline{M}_d$ coarsely represents $\overline{\mathcal{M}}_d$.)

Level structures: there exists a finite cover $\pi : M_d^{lev} \twoheadrightarrow M_d$ over which there is a universal family. M_d^{lev} is smooth.

Comparison: $\mathcal{A}_g$

A $\quad A[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$ Weil pairing

$0 \leq p \nmid n$ level n structure

symplectic level-n structure

$\mathcal{A}_g[n]$ nonsing. q-proj. var.

$\mathcal{A}_g[n] / Sp(2g, \mathbb{Z}/n\mathbb{Z}) \cong \mathcal{A}_g$.

$\Gamma_\ell := \{ g \in \tilde{O}(\Lambda_d) \mid g \equiv \mathrm{id}\ (\ell) \}$

$\ell >> 0$: Γ_ℓ torsion-free, finite index, acts freely.

N_d : saw already. marked

Over this, univ. family.

$\tilde{O}(\Lambda_d)$ acts on N_d; action lifts to the universal family. Uses another part of Global Torelli: any Hodge isometry mapping ℓ to ℓ' can be lifted uniquely to an isom. $X \xrightarrow{\sim} X'$.

$M_d^{lev} := \Gamma_\ell \backslash N_d$ nonsing.

parametrizes polarized K3's (X, L)

with an isom. $H^2(X, \mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{\sim} \Lambda \otimes \mathbb{Z}/\ell\mathbb{Z}$

compatible with the pairing and mapping L to $e_1 + d f_1$.

(Alternative: $H^2(X, \mathbb{Z}/\ell\mathbb{Z})_{prim} \xrightarrow{\sim} \Lambda_d \otimes \mathbb{Z}/\ell\mathbb{Z}$.)

" ... with a $\Lambda/\ell\Lambda$-level structure"

$[H/PGL]$ D-M stack $\mathcal{M}_d$

$\underbrace{[M_d^{lev}/G]}_{\text{D-M stack}}$ $G := \tilde{O}(\Lambda_d)/\Gamma_\ell$.

The two stacks are isomorphic.

"Applications"

$f : X \to S$ polarized K3's S smooth conn.

cx. var.

$R^2 f_* \mathbb{Z}$

$\rho : \pi_1(S,t) \to O(H^2(X_t,\mathbb{Z}))$ monodromy

image : monodromy group.

Zariski closure in $O(H^2(X_t,\mathbb{Q}))$ is "algebraic monodromy group".

* Any (X_0, L_0) sits in a family s.th. $\mathrm{Im}(\rho)$ is finite index in the subgroup of $O(H^2(X_0,\mathbb{Z}))$ that fixes L_0

(this subgroup is isom. to $\tilde{O}(\Lambda_d)$).

* Better: $\exists S$ for which $\mathrm{Im}(\rho)$ equals $\tilde{O}(\Lambda_d)$.

the whole

$\mathrm{im}\,\beta)$

* Deligne: $H^2(X_+, \mathbb{Q})$

$$= \mathrm{im}\left(H^2(\bar{X}, \mathbb{Q}) \to H^2(X, \mathbb{Q}) \right)$$

where $X \subset \bar{X}$ arbitrary smooth projective compactification.

Deligne torus: $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_{m,\mathbb{C}}$

$\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$ for any $\mathbb{R}$-alg. A

V real v.sp.: $\{\text{H.S. on } V\} \leftrightarrow \{\rho : \mathbb{S} \to GL(V_{\mathbb{R}})\}$ morphism of real alg. gps.

$\mathbb{U} \subset \mathbb{S}$: kernel of norm $\mathrm{Nm}: \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$

$\mathbb{U}(\mathbb{R}) = \{ z \mid |z| = 1 \} \subset \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$.

V $\mathbb{Q}$-v.sp. with polarizable Hodge str

$$(\Leftrightarrow \quad \rho : \mathbb{S} \to GL(V_{\mathbb{R}}))$$

Hodge group: $Hdg(V)$ smallest alg. subgp. of $GL(V)/\mathbb{Q}$ with $\rho(\mathbb{U}(\mathbb{R})) \subset Hdg(V)(\mathbb{R})$.

Mumford-Tate group: $MT(V)$ ..... $\rho(\mathbb{S}(\mathbb{R})) \subset MT(V)(\mathbb{R})$.

A subspace $W \subset \bigoplus_i V^{\otimes n_i} \otimes V^{* \otimes m_i}$ is a subHS $\Leftrightarrow$ W is preserved by $MT(V)$.

<u>Thm.</u> Any $X \to S$ as above, $\exists\, S' \subset S$, countable union of proper closed subvarieties, s.th. $\forall t \in S \setminus S'$: $MT(H^2(X_t, \mathbb{Q}))$ is constant and contains a finite index subgroup of $\overline{\Gamma_t(s)}$ (and hence

a finite index subgroup of $\mathrm{Im}(\rho)$ (and hence the identity component).

k fin. gen. field of char. 0 $\quad X$ sm. proj. var. $/k$

$$\rho_\ell : \mathrm{Gal}(\bar{k}/k) \to GL(H^2_{\acute{e}t}(X \otimes_k \bar{k}, \mathbb{Q}_\ell(1)))$$

$\overline{\mathrm{Im}(\rho_\ell)}$: ℓ-adic alg. monodromy group.

<u>Thm.</u>

$$\overline{\mathrm{Im}(\rho_\ell)}^{\circ} = MT(H^2(X_{\mathbb{C}}, \mathbb{Q}(1))) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$