Arithmetic invariants of supersingular abelian varieties

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Set-up

- Let p be a prime and $k = \overline{\mathbb{F}}_p \supseteq \mathbb{F}_q \supseteq \mathbb{F}_p$.
- An elliptic curve $E/\mathbb{F}_q$ is SUPERSINGULAR if $E[p](k) = \{0\}$ (and ORDINARY otherwise).
- We can classify g-dimensional abelian varieties up to **isomorphism** ($\simeq$) or up to **isogeny** ($\sim$).
- A g-dimensional abelian variety X is SUPERSINGULAR if $X \sim_k E^g$ with E a supersingular elliptic curve, and SUPERSPECIAL if $X \simeq_k E^g$.
- A POLARISATION $\lambda : X \to X^\vee$ is an isogeny induced from an ample line bundle on X_k.
- A polarisation is PRINCIPAL if it is an isomorphism. Then (X, λ) is a PPAV = principally polarised abelian variety.

Moduli space

Let $\mathcal{A}_g$ be the moduli space (in characteristic p) of g-dimensional ppAVs. $\mathcal{A}_g$ is irreducible of dimension $\frac{g(g+1)}{2}$.

Let $\mathcal{S}_g$ be the supersingular locus, so

$$\mathcal{S}_g(k) = \{(X, \lambda) \in \mathcal{A}_g(k) : X \text{ is supersingular}\}.$$

$\mathcal{S}_g$ has dimension $\lfloor \frac{g^2}{4} \rfloor$. The number of ireducible components is

$$H_g(p, 1) \text{ if } g \text{ is odd, and } H_g(1, p) \text{ is } g \text{ is even.}$$

Theorem 1 (Ibukiyama-K.-Yu)

$\mathcal{S}_g$ is geometrically irreducible if and only if one of the following holds:

- $g = 1$, $p \in \{2, 3, 5, 7, 13\}$;
- $g = 2$, $p \in \{2, 3, 5, 7, 11\}$;
- $(g, p) = (3, 2)$ or $(4, 2)$.

Geometric structures on $\mathcal{A}_g$ and $\mathcal{S}_g$

For any (X,λ), consider its p-torsion group scheme $(X,\lambda)[p]$ and its p-divisible group $(X,\lambda)[p^\infty] = \varinjlim_n (X,\lambda)_{p^n}$, up to isomorphism ($\simeq$) and up to isogeny ($\sim$).

1. The geometric isogeny class of $(X,\lambda)[p^\infty]$ uniquely determines a Newton polygon. $\Rightarrow$ NEWTON STRATIFICATION of $\mathcal{A}_g$.
   Fact: Supersingular ppAVs have a unique p-divisible group up to k-isogeny $\Leftrightarrow$ unique Newton polygon $(\frac{1}{2},\ldots,\frac{1}{2})$.
   So $\mathcal{S}_g$ is a Newton stratum.
2. The a-NUMBER of $(X,\lambda) \in \mathcal{A}_g(k)$ is
   $$a(X) = \dim_k \mathrm{Hom}(\alpha_p, X).$$
   It depends on $X[p]/\simeq$. If $(X,\lambda) \in \mathcal{S}_g(k)$ then $1 \leq a(X) \leq g$.
   Also X is superspecial $\Leftrightarrow a(X) = g$.
   $\Rightarrow$ a-NUMBER STRATIFICATION of $\mathcal{A}_g$ and $\mathcal{S}_g$.

Geometric structures on $\mathcal{A}_g$ and $\mathcal{S}_g$

3. Canonical filtration of $X[p]$ indexed by elementary sequence φ $\Rightarrow$ Ekedahl-Oort stratification of $\mathcal{A}_g$ and $\mathcal{S}_g$, i.e.

$$\mathcal{S}_g = \sqcup_\varphi (\mathcal{S}_\varphi \cap \mathcal{S}_g).$$

Combinatorial criterion (Chai-Oort) says when $\mathcal{S}_\varphi \subseteq \mathcal{S}_g$; supersingular Ekedahl-Oort strata $\varphi \in \Phi^{ss}$.

Now from $X[p^\infty]/\sim$ and $X[p]/\simeq$ to $X[p^\infty]/\simeq$.
The central leaf through $x = (X, \lambda) \in \mathcal{A}_g(k)$ is

$$\mathcal{C}(x) = \{(X', \lambda') \in \mathcal{A}_g(k) : (X', \lambda')[p^\infty] \simeq (X, \lambda)[p^\infty]\}.$$

Oort showed foliation structure on $\mathcal{S}_g$:
every irreducible component of $\mathcal{S}_g$ is isomorphic, up to a finite morphism, to (central leaf) x (isogeny leaf).

Chai showed $\mathcal{C}(x)$ is finite $\Leftrightarrow$ $x = (X, \lambda)$ is supersingular.

Automorphisms of $(X, \lambda) \in \mathcal{S}_g(k)$

Chai and Oort showed that for generic point $\eta = (X_\eta, \lambda_\eta)$ of $\mathcal{A}_g$ we have $\mathrm{End}(X_\eta) = \mathbb{Z}$ and $\mathrm{Aut}(X_\eta, \lambda_\eta) = \{\pm 1\}$.

What about $\mathcal{S}_g$?

Oort's conjecture: Let $g \geq 2$, p prime. Every generic g-dimensional supersingular ppAV $(X, \lambda) \in \mathcal{S}_g(k)$ has

$$\mathrm{Aut}(X, \lambda) = \{\pm 1\}.$$

There are counterexamples for $(g, p) = (2, 2)$ (Ibukiyama) and $(g, p) = (3, 2)$ (Oort).

Status of Oort's conjecture (OC)

$\boldsymbol{g} = 2$: OC holds for all $p \neq 2$.

K-Pries showed that for $p \geq 3$, the proportion of $\mathcal{S}_2(\mathbb{F}_{p^r})$ with $\mathrm{Aut} \neq \{\pm 1\}$ tends to zero as $r \to \infty$;
it suffices to consider *superspecial curves*.

Ibukiyama determined the mass (= weighted count) of principal polarisations of supersingular surfaces to get the same result.

Status of Oort's conjecture (OC)

$\boldsymbol{g} = 3$: OC holds for all $p \neq 2$ (K-Yobuko-Yu).

Idea: From minimal isogeny $\psi : (\widetilde{X}, \widetilde{\lambda}) \to (X, \lambda)$ compare Dieudonné modules $(M, \langle , \rangle)$ of (X, λ) and $(\widetilde{M}, \langle , \rangle)$ of $(\widetilde{X}, \widetilde{\lambda})$. Then $\mathrm{End}(X) \otimes \mathbb{Z}_p \simeq \mathrm{End}(M)$ and

$$\mathrm{Aut}(X, \lambda) = \mathrm{Aut}(\widetilde{X}, \widetilde{\lambda}) \cap \mathrm{Aut}(M, \langle , \rangle).$$

Furthermore,

$$\mathrm{Aut}(M, \langle , \rangle) \subseteq \mathrm{Aut}(\widetilde{M}, \langle , \rangle) \simeq \{A \in \mathrm{GL}_3(\mathcal{O}_p) : A^* A = \mathbb{I}_3\},$$

for $\mathcal{O}_p$ the maximal order in division quaternion algebra $D_p/\mathbb{Q}_p$ with uniformiser Π.Have $\mathrm{Aut}(X, \lambda) \xrightarrow{\text{mod } p} \dots \xrightarrow{\text{mod } \Pi} \{\pm 1\}$. This is injective when $p \geq 5$, since

$$\ker \subseteq (1 + \Pi \mathrm{Mat}_3(\mathcal{O}_p))_{\mathrm{tors}} = \{1\}.$$

Status of Oort's conjecture (OC)

$\boldsymbol{g} = 4$: OC holds for $p \neq 2$ (Dragutinovic), all p (K-Yu).

Dragutinovic showed for $p > 2$ that every irreducible component of M_4^{ss} has trivial generic automorphism group.

K-Yu showed the result for all p via explicit computations, refining ideas from $g = 3$ by using further reductions (mod $V^3 = pV$) of $\mathrm{Aut}(M, \langle , \rangle)$.

Status of Oort's conjecture (OC)

$\boldsymbol{g} \geq 5$: OC holds for even g and $p \geq 5$ (K-Yu).

Recall supersingular EO strata $\Phi^{ss} = \{\varphi : \mathcal{S}_\varphi \subseteq \mathcal{S}_g\}$.
Consider $\mathcal{S}_g^{eo} = \bigcup_{\varphi \in \Phi^{ss}} \mathcal{S}_\varphi$.
Then $\mathcal{S}_g^{eo} = \overline{\mathcal{S}_{\varphi_{\max}}}^{\mathrm{Zar}}$ for $\varphi_{\max} = (0, \ldots, 0, 1, 2, \ldots, \lfloor \frac{g}{2} \rfloor)$.

Also $(X, \lambda) \in \mathcal{S}_g^{eo} \Leftrightarrow \exists\, 0 \leq c \leq \lfloor \frac{g}{2} \rfloor$, and polarisation μ on E^g such that $\ker \mu \simeq \alpha_p^{2c}$, and isogeny $\rho : (E^g, \mu) \to (X, \lambda)$.
There is a finite surjection

$$\mathrm{pr} : \coprod_{\mu : \ker \mu \simeq \alpha_p^{\lfloor \frac{g}{2} \rfloor}} \mathcal{X}_\mu \to \mathcal{S}_g^{eo},$$

where $\mathcal{X}_\mu = \left\{ \rho : (E^g, \mu) \to (X, \lambda) \text{ of degree } p^{\lfloor \frac{g}{2} \rfloor} \right\}$ is a Lagrangian variety, parametrising maximal isotropic subspaces of $(\overline{M}^{\diamond}, \psi_{\overline{M}^{\diamond}})$ over $\mathbb{F}_{p^2}$, obtained from Dieudonné module of (E^g, μ).

Status of Oort's conjecture (OC)

$\boldsymbol{g} \geq 5$ continued:
Using the $\mathcal{X}_\mu$, we can stratify $\mathcal{S}_g^{eo}$ by endomorphism ring.

Theorem 2 (K.-Yu)

The stratum with minimal endomorphism ring is open and dense. When g is even and $p \geq 5$, any (X, λ) in it has $\mathrm{Aut}(X, \lambda) = \{\pm 1\}$.

$\Rightarrow$ Generic point of $\mathcal{S}_{\varphi_{\max}}$ has automorphism group $\{\pm 1\}$.

Theorem 3 (K.-Yu)

For $\ell \neq p$, the ℓ-adic Hecke correspondences induce transitive action on irreducible components of $\mathcal{S}_g$.

$\Rightarrow$ Every irreducible component of $\mathcal{S}_g$ contains an irreducible component of $\mathcal{S}_{\varphi_{\max}}$.
$\Rightarrow$ Generic point of every irreducible component of $\mathcal{S}_g$ has automorphism group $\{\pm 1\}$, i.e. OC.

Gauss problem for central leaves

Theorem 4 (Ibukiyama-K.-Yu)

Let $x = (X, \lambda) \in \mathcal{S}_g(k)$. Then $\#\mathcal{C}(x) = 1$ if and only if one of the following holds:

- $g = 1$, $p \in \{2, 3, 5, 7, 13\}$;
- $g = 2$, $p \in \{2, 3\}$;
- $g = 3, p = 2$ and $a(X) \geq 2$.

Fact 1: Every central leaf is contained in a unique EO-stratum (hence a-number stratum).
Fact 2: Every $x = (X, \lambda) \in \mathcal{S}_g(k)$ has a unique MINIMAL ISOGENY $\psi : \widetilde{X} \to X$ with $\widetilde{X} \simeq E^g$ superspecial but not necessarily principally polarised, so that $\mathcal{C}(x) \twoheadrightarrow \mathcal{C}(\widetilde{x})$, for $\widetilde{x} = (\widetilde{X}, \widetilde{\lambda} = \psi^* \lambda)$.
Fact 3: For any superspecial $\widetilde{x} = (\widetilde{X}, \widetilde{\lambda})$ we know

$$\mathrm{Mass}(\mathcal{C}(\widetilde{x})) = \sum_{x' \in \mathcal{C}(\widetilde{x})} \frac{1}{|\mathrm{Aut}(x')|}.$$

Proof sketch of Theorem 4

$\boldsymbol{g} = 1$: Number of irreducible components of $\mathcal{S}_1$ = number of supersingular elliptic curves up to isomorphism; determined by Deuring and Eichler. Also there is a unique central leaf, so $\#\mathcal{C}(x) = 1 \Leftrightarrow \#$ irr. components of $\mathcal{S}_1 = 1$.

$\boldsymbol{g} = 2, 3$: Minimal isogeny $\psi : \widetilde{X} \to X$ of X only depends on $a(X)$. For each $a(X) \in \{1, 2, \ldots, g\}$ we compute $\mathrm{Mass}(\mathcal{C}(x))$ from suitable $\mathrm{Mass}(\mathcal{C}(\widetilde{x}))$ via ψ. We even find $\#\mathcal{C}(x)$ explicitly.

$\boldsymbol{g} = 5$: Follows from $\mathcal{C}(x) \twoheadrightarrow \mathcal{C}(\widetilde{x})$ since all $\#\mathcal{C}(\widetilde{x}) > 1$.

Proof sketch of Theorem 4

$\boldsymbol{g} = 4$: Need finer EO data.

Proposition (Ibukiyama-K.-Yu)

$$\mathcal{S}_4 = (\mathcal{S}_{(0,1,2,3)} \cap \mathcal{S}_4) \sqcup (\mathcal{S}_{(0,1,1,2)} \cap \mathcal{S}_4) \sqcup \mathcal{S}_{(0,0,0,0)} \sqcup \mathcal{S}_{(0,0,0,1)} \sqcup \mathcal{S}_{(0,0,1,1)} \sqcup \mathcal{S}_{(0,0,1,2)}$$

For each of these six EO strata, we determine suitable

$$\mathcal{C}(x) \twoheadrightarrow \mathcal{C}(\widetilde{x})$$

and say enough about $\mathcal{C}(\widetilde{x})$ to conclude $\#\mathcal{C}(x) > 1$.

What is next?

Many questions remain open:

- Interaction of (esp. Newton and EO) stratifications in dimension $g \geq 5$?
- Intersection of strata with Torelli locus?
- Oort's conjecture for odd $g \geq 5$ and for $p = 2, 3$?
- Mass formulae for other abelian varieties (dimension $g \geq 4$, non-supersingular)?
- Cardinalities of central leaves?
- Endomorphisms: which endomorphism rings occur? (K-Tamagawa-Yu, in progress)
- ...

Thank you for your attention!