Random matrix ensemble for the level statistics of many-body localization

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Universality of level statistics

Hamiltonian $H$

- Spectrum
  
  $H \psi_i \rangle = E_i \psi_i \rangle$  
  $E_{i+1} \geq E_i$

- Density of states
  
  $\rho(E) = \sum_i \delta(E - E_i)$

- Short-range fluctuations show universal statistics
  
  - Poissonian: integrable systems
  - Wigner-Dyson: ergodic systems ($\beta = 1, 2, 4$)

- Dyson index $\beta$: degrees of freedom per matrix element
  
  - Real entries: $\beta = 1$
  - Imaginary entries: $\beta = 2$
  - Quaternionic entries: $\beta = 4$
Random matrix theory

- Universality of level statistics is studied in *random matrix theory*
- Philosophy: gain understanding by studying tractable random matrices with similar statistics

**Example: Gaussian Orthogonal Ensemble (GOE)**

Let \( Y \) be an \( n \times n \) matrix of independent standard Gaussians \( N[0, 1] \) used to form \( X = \frac{1}{2}(Y + Y^T) \)

- Displays Wigner-Dyson (\( \beta = 1 \)) level statistics
- Eigenvalue density

\[
\rho(e_1, \ldots, e_n) = C_n \prod_{i<j} |e_i - e_j| \prod_{i=1}^{n} e^{-\frac{1}{2}e_i^2}
\]
Gaussian random matrix ensembles

- Three Gaussian random matrix ensembles:
  - Gaussian Orthogonal Ensemble (GOE): $\beta = 1$
  - Gaussian Unitary Ensemble (GUE): $\beta = 2$
  - Gaussian Symplectic Ensemble (GSE): $\beta = 4$

- Named after type of transformations used to diagonalize

- Eigenvectors uniformly distributed w.r.t. the uniform Haar measure $\rightarrow$ ensemble invariance under basis transformations

- Eigenvalue density

\[
\rho(e_1, \ldots, e_n) = C_{n,\beta} \prod_{i < j} |e_i - e_j|^\beta \prod_{i=1}^{n} e^{-\frac{\beta}{2} e_i^2}
\]
Gaussian beta ensemble


- Matrix model with eigenvalue p.d.f. of the Gaussian ensembles for generic $\beta > 0$

\[ T = \frac{1}{\sqrt{\beta}} \begin{bmatrix} a_n & b_{n-1} \\ b_{n-1} & a_{n-1} & b_{n-2} \\ b_{n-2} & a_{n-2} & b_{n-3} \\ \vdots & \vdots & \vdots \\ b_2 & a_2 & b_1 \\ b_1 & a_1 \end{bmatrix} \]

with $a_i \sim N[0, 1]$ and $b_i \sim \chi_i \beta$

- Interpolates between Poissonian ($\beta \to 0$) and Wigner-Dyson ($\beta = 1, 2, 4$)
Overview

- Proposal: generalize Wigner-Dyson level statistics beyond $\beta = 1, 2, 4$ through the Gaussian $\beta$ ensemble

- Motivation: observation of intermediate level statistics for ‘non-maximally chaotic’ systems

- Investigate physical relevance by checking against level statistics across the many-body localization transition

Related references (selection)

- Serbyn, Moore, PRB 93, 041424 (2016)

- Kjäll, PRB 97, 035163 (2018)

Physical model

- Spin chain with random onsite disorder:

\[ H = \sum_{i=1}^{L} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z) + \sum_{i=1}^{L} h_i S_i^z \]

- Disorder \( h_i \) sampled from uniform distribution over \([-W, W]\)

- Periodic boundary conditions \( S_{L+1}^{x,y,z} \equiv S_1^{x,y,z} \)

- Total zero magnetization \( \sum_i S_i^z = 0 \)

- Focus: \( L = 16 \), middle of the spectrum

- Crossover from thermal to many-body localized phase at \( 2 \lesssim W \lesssim 4 \) [Luitz, Laflorencie, Alet, PRB 91 081103 (2015)]
**Ratio of consecutive level spacings**

- **Focus:** distribution of ratio of consecutive level spacings

\[ r_i = \min \left( \frac{s_{i+1}}{s_i}, \frac{s_i}{s_{i+1}} \right), \quad s_i = E_{i+1} - E_i \]

- Independent of local density of states, no unfolding required

- **Ref:** Atas, Bogomolny Giraud, Roux, PRL 110, 084101 (2013)

![Graph showing the comparison of Poissonian and Wigner-Dyson distributions](image.png)

- Random matrix ensemble for the level statistics of MBL
Comparison with level statistics

- Near-perfect agreement over full range
- Similar fit quality for level spacing distribution
- Similar results for $L = 12, 14$
- Next step: extension to $\beta \in [1, 2]$
- Next step: longer-range correlations

\[\begin{align*}
W = 2 & \ (\beta \approx 0.86) \\
W = 3 & \ (\beta \approx 0.23) \\
W = 4 & \ (\beta \approx 0.05) \\
W = 5 & \ (\beta \approx 0.02)
\end{align*}\]
Next step: extension to beta > 1

- $H \rightarrow H + H'$,

$$H' = \sum_{i=1}^{L} \vec{S}_i \cdot (\vec{S}_{i+1} \times \vec{S}_{i+2}),$$

- Breaks time-reversal symmetry ($\beta = 2$)
- Near-perfect agreement
- Crossing of $\beta = 1$
Next step: longer-range correlations

- Higher-order spacing ratios

\[ r_i^{(n)} = \min \left( \frac{E_{i+2n} - E_{i+n}}{E_{i+n} - E_i}, \frac{E_{i+n} - E_i}{E_{i+2n} - E_{i+n}} \right). \]

- Qualitative agreement up to \( \sim 5 \) level spacings
Conclusions and outlook

• Conclusions:
  • Proposed generalized Wigner-Dyson level statistics through the Gaussian $\beta$ ensemble
  • Good agreement with the level statistics across the many-body localization transition

• Outlook:
  • What does it learn us about many-body localization?
  • Show intermediate level statistics universal characteristics?
  • Does the idea translate to e.g. the spectral statistics of entanglement spectra?

Thank you!