Quadrangulating the Sphere

Lowell Abrams
The George Washington University
Washingtion DC, USA

Dan Slilaty
Wright State University
Dayton Ohio, USA

Preamble

spherical quadrangulation:

a graph embedding in the sphere
in which every face has boundary length 4.

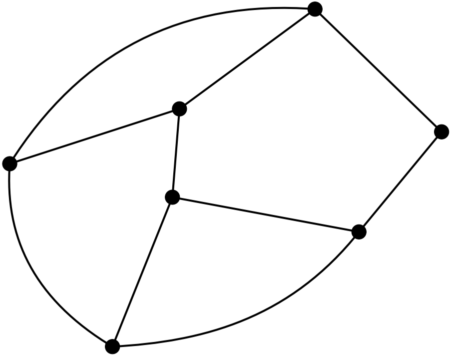
What we accomplished

Construction of every spherical quadrangulation
having order n rotational symmetry, $n \geq 3$,
subject to a constraint on degrees of vertices.

Graph Embeddings

graph embeddings and dual embeddings

Embeddings must be *cellular*: the complement of the graph is a disjoint union of contractible regions ("faces")

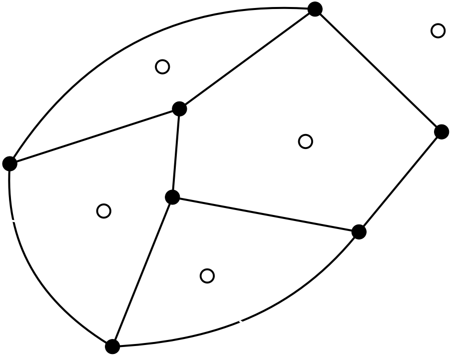


A graph embedding is a triple of sets: (V, E, F)

Write $G = (V, E)$ and $G^* = (F, E)$

graph embeddings and dual embeddings

Embeddings must be *cellular*: the complement of the graph is a disjoint union of contractible regions ("faces")

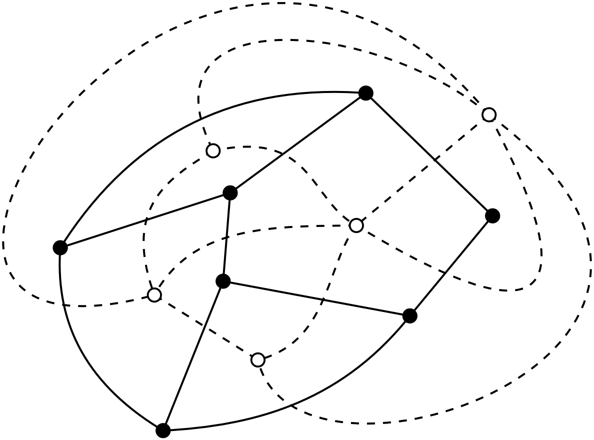


A graph embedding is a triple of sets: (V, E, F)

Write $G = (V, E)$ and $G^* = (F, E)$

graph embeddings and dual embeddings

Embeddings must be *cellular*: the complement of the graph is a disjoint union of contractible regions ("faces")



A graph embedding is a triple of sets: (V, E, F)

Write $G = (V, E)$ and $G^* = (F, E)$

Curvature Without Angle Measure

Euler's formula

For any cellular graph embedding (V, E, F)
in a topological surface of genus g,

$$|V| - |E| + |F| = 2 - 2g$$

Proof.

- Reduce to $|V| = 1$ by contracting edges

∎

Euler's formula

For any cellular graph embedding (V, E, F)
in a topological surface of genus g,

$$|V| - |E| + |F| = 2 - 2g$$

Proof.

- Reduce to $|V| = 1$ by contracting edges
- Reduce to $|F| = 1$ by deleting edges

$\blacksquare$

Euler's formula

For any cellular graph embedding (V, E, F)
in a topological surface of genus g,

$$|V| - |E| + |F| = 2 - 2g$$

Proof.

- Reduce to $|V| = 1$ by contracting edges
- Reduce to $|F| = 1$ by deleting edges
- Observe that theorem holds for embeddings with $|V| = |F| = 1$

$\blacksquare$

flat configurations

Suppose G and G^* are simple. Given

$$\begin{aligned} V_i &:= \{\text{vertices of degree } i\} \\ F_i &:= \{\text{faces of boundary length (“codegree”) } i\} \end{aligned}$$

we have

$$2|E| = \sum_{i\geq 3} i|V_i| \text{ and } 2|E| = \sum_{i\geq 3} i|F_i|$$

flat configurations

Suppose G and G^* are simple. Given

$$\begin{aligned} V_i &:= \{\text{vertices of degree } i\} \\ F_i &:= \{\text{faces of boundary length (“codegree”) } i\} \end{aligned}$$

we have

$$2|E| = \sum_{i \geq 3} i|V_i| \quad \text{and} \quad 2|E| = \sum_{i \geq 3} i|F_i|$$

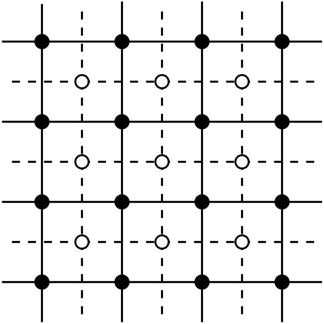
Thus

$$|V| - |E| + |F| = \sum_{i \geq 3} |V_i| - \left(\frac{1}{4} \sum_{i \geq 3} i|V_i| + \frac{1}{4} \sum_{i \geq 3} i|F_i| \right) + \sum_{i \geq 3} |F_i|$$

Note: vertices of degree 4 and faces of codegree 4
make no contribution, *i.e.* **have no impact on genus**.

flat configurations

the $\{4, 4\}$ grid



curvature vertices and balancing

curvature vertex: a vertex with degree other than 4

$$|V| - |E| + |F| = \sum_{i \geq 3} |V_i| - \left(\frac{1}{4} \sum_{i \geq 3} i|V_i| + \frac{1}{4} \sum_{i \geq 3} i|F_i| \right) + \sum_{i \geq 3} |F_i|$$

example:

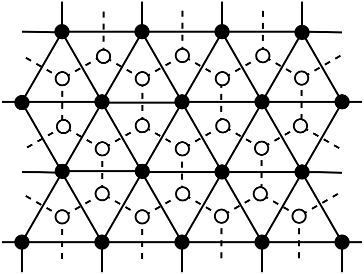
contribution from degree 6 vertices and codegree 3 faces:

$$|V_6| - \frac{6}{4}|V_6| - \frac{3}{4}|F_3| + |F_3| \quad = \quad -\frac{1}{2}|V_6| + \frac{1}{4}|F_3|$$

So, if $|F_3| = 2|V_6|$, then have neutral contribution to genus

curvature vertices and balancing

the $\{6, 3\}$ grid



$$3|F_3| = 2|E| = 6|V_6|$$

curvature vertices and balancing

example:

suppose genus is 0, all vertices have degree 3 or 4,
and all faces have codegree 4, 5 or 6

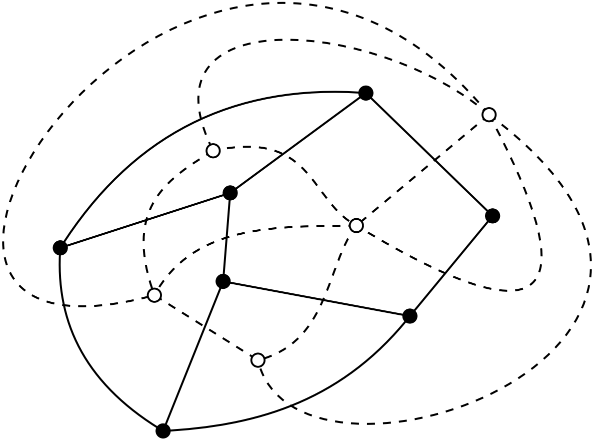
$$\begin{aligned} 2 &= |V| - |E| + |F| \\ &= \tfrac{1}{4}|V_3| - \tfrac{1}{4}|F_5| - \tfrac{1}{2}|F_6| \end{aligned}$$

$$\textit{i.e.}\ |F_5| + 2|F_6| = |V_3| - 8$$

Need 8 vertices of degree 3 ("positive curvature") to get cube,
then additional vertices of degree 3
to balance the faces of codegree 5 or 6 ("negative curvature")

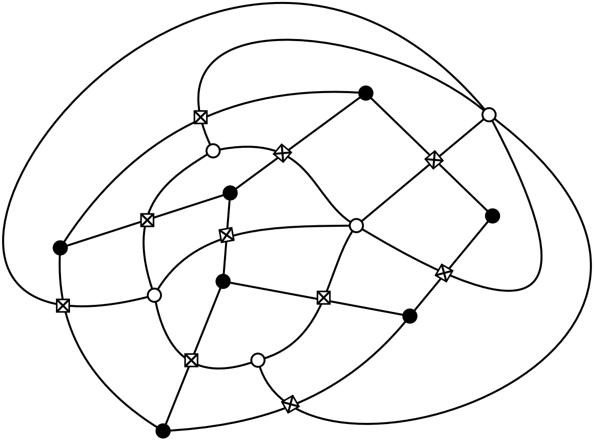
curvature vertices and balancing

"overlay graph" combines primal and dual graphs



curvature vertices and balancing

"overlay graph" combines primal and dual graphs

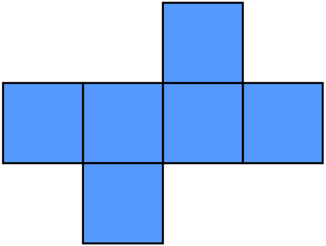


now all faces are square and graph is bipartite.
all curvature is concentrated at ○ and ● vertices,
all ⊠ vertices are degree four

Nets

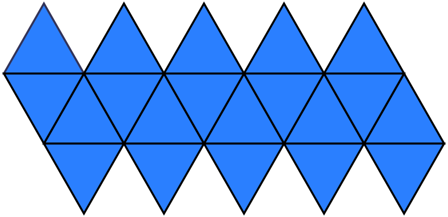
cutting polyhedra open

"a *net* of a polyhedron is an arrangement
of non-overlapping edge-joined polygons in the plane
which can be folded (along edges)
to become the faces of the polyhedron." (from Wikipedia)



cutting polyhedra open

"a *net* of a polyhedron is an arrangement
of non-overlapping edge-joined polygons in the plane
which can be folded (along edges)
to become the faces of the polyhedron." (from Wikipedia)



cutting polyhedra open

"a *net* of a polyhedron is an arrangement
of non-overlapping edge-joined polygons in the plane
which can be folded (along edges)
to become the faces of the polyhedron." (from Wikipedia)

G. C. Shephard (1975): Does every convex polyhedron have a net?

why nets?

- Nets prove existence.

why nets?

- Nets prove existence.
- Well-chosen nets can display important structure, such as symmetries.

why nets?

- Nets prove existence.
- Well-chosen nets can display important structure, such as symmetries.
- Nets provide a model for physical construction.

why nets?

- Nets prove existence.
- Well-chosen nets can display important structure, such as symmetries.
- Nets provide a model for physical construction.
- Every example constructed in this work has a net.

The Main Result

some background

Servatius and Servatius, 1995:
: use overlay graph to classify self-dual spherical embeddings, along with corresponding part-preserving cellular automorphisms of the overlay graphs

Graver and Hartung, 2014:
: use overlay graph and nets to give more concrete classification of self-dual spherical embeddings; require graphs have four degree-3 vertices and remaining vertices all of degree 4

This Work:
: use overlay graph and nets; concrete; not necessarily self-dual; more allowed degree sequences; require rotational symmetry

some specifics

What we accomplished

Construction of every spherical quadrangulation
having order n rotational symmetry, $n \geq 3$,
and, subject to that condition,
having as few curvature vertices as possible.

some specifics

What we accomplished

Construction of every spherical quadrangulation
having order n rotational symmetry, $n \geq 3$,
and, subject to that condition,
having as few curvature vertices as possible.

- Without loss of generality (overlay graph!) assume the embedded graph is bipartite, and all curvature vertices are in the same part of the bipartition.

some specifics

What we accomplished

Construction of every spherical quadrangulation
having order n rotational symmetry, $n \geq 3$,
and, subject to that condition,
having as few curvature vertices as possible.

- Without loss of generality (overlay graph!) assume the embedded graph is bipartite, and all curvature vertices are in the same part of the bipartition.
- Must have two vertices of degree n that will serve as the poles of the rotational symmetry

some specifics

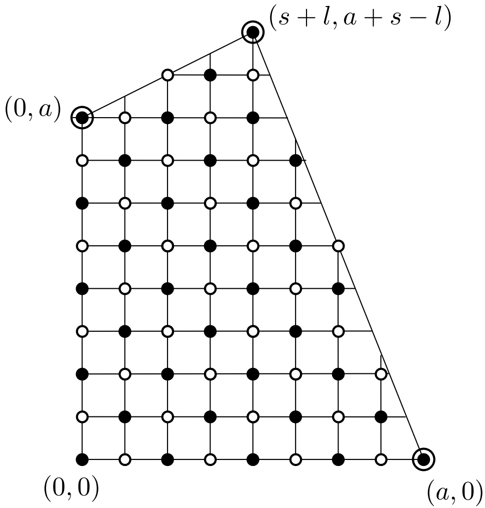
What we accomplished

Construction of every spherical quadrangulation
having order n rotational symmetry, $n \geq 3$,
and, subject to that condition,
having as few curvature vertices as possible.

- Without loss of generality (overlay graph!) assume the embedded graph is bipartite, and all curvature vertices are in the same part of the bipartition.
- Must have two vertices of degree n that will serve as the poles of the rotational symmetry
- Must have $2n$ degree-3 vertices to balance those:

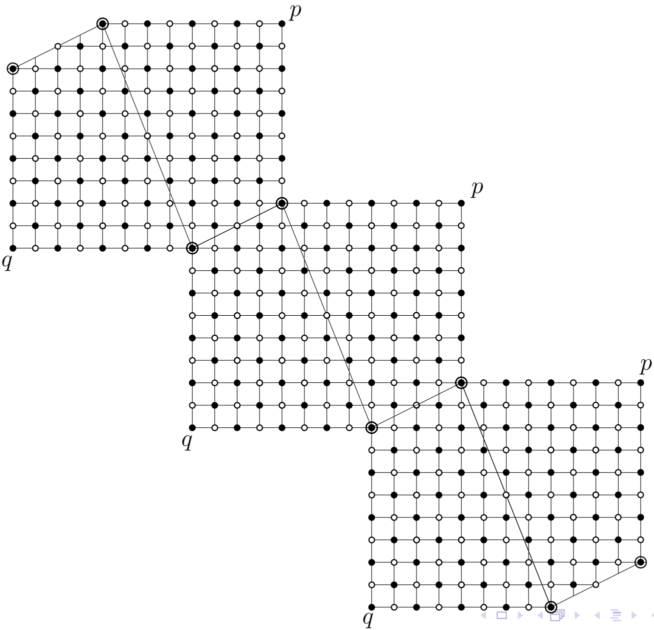
$$2 = (|V_3| + |V_n|) - \frac{1}{4}\left(3|V_3| + n|V_n|\right) = \frac{1}{4}|V_3| + 2 - \frac{1}{2}n$$

two kinds of panel, two kinds of net

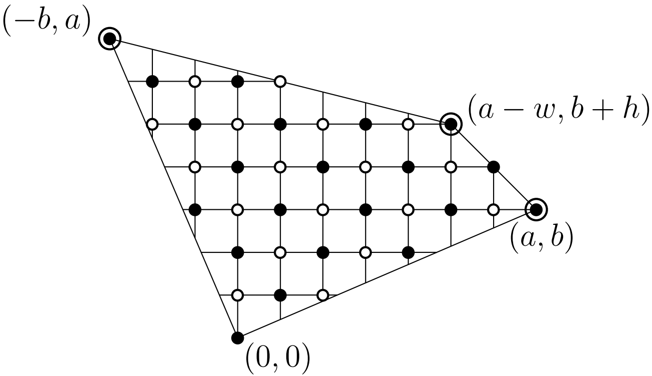


$$a \in 2\mathbb{Z}_{>0}, \ s \in \mathbb{Z}_{\geq 0}, \ l \in \{0, 1, \ldots, a-1\}$$

two kinds of panel, two kinds of net



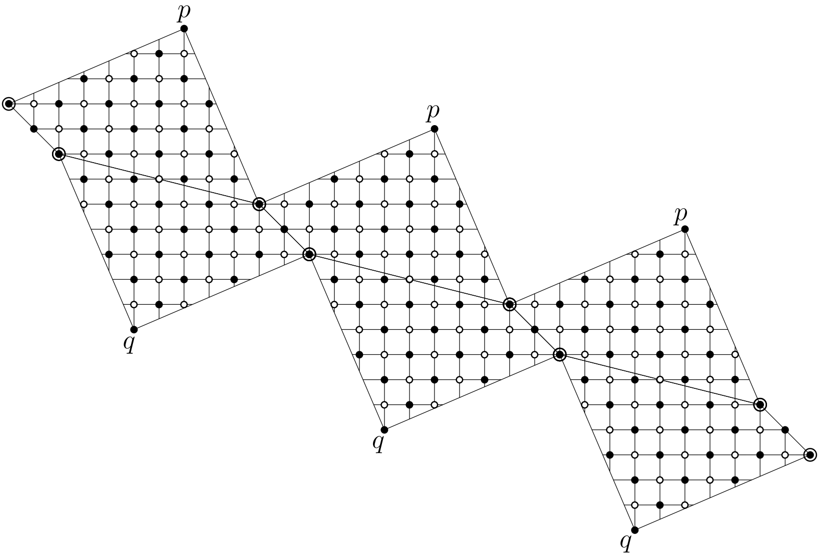
two kinds of panel, two kinds of net



$$a, b \in \mathbb{Z}_{>0}, \quad a \equiv_2 b, \qquad h, w \in \mathbb{Z}_{\geq 0}, \quad h \equiv_2 w, \;\; h + w \neq 0$$

$$-\frac{h}{w} \leq \frac{b-a}{b+a}, \quad -\frac{a}{b}(a-w) \leq b + h$$

two kinds of panel, two kinds of net



the main result

Theorem (A. and Slilaty, ∼Covid)

Every spherical quadrangulation
with order n rotational symmetry, $n \geq 3$, and
with as few curvature vertices as possible
can be constructed
using one of the two constructions shown above

the main result

Theorem (A. and Slilaty, ∼Covid)

Every spherical quadrangulation
with order n rotational symmetry, $n \geq 3$, and
with as few curvature vertices as possible
can be constructed
using one of the two constructions shown above

Corollary

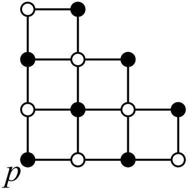
Every example mentioned in the theorem
has a hemisphere-flip or glide-reflection symmetry

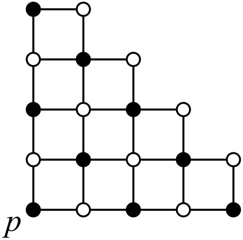
Examples!!

Growing Polar Caps

standard k-balls

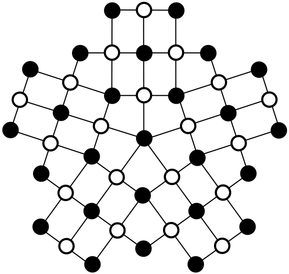
the standard 3-wedge and the standard 4-wedge
(cut from the $\{4,4\}$ grid)

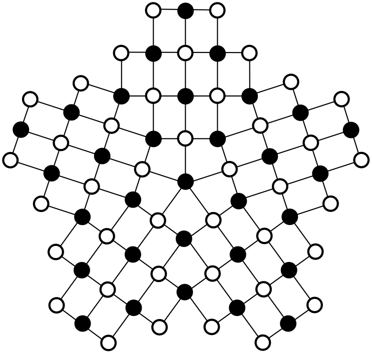




standard k-balls

the standard 3-ball of order 5 and the standard 4-ball of order 5

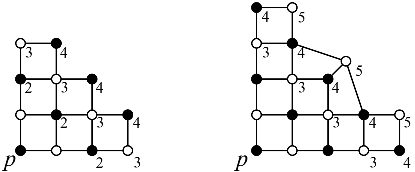




standard k-balls

G: a spherical quadrangulation having
order-n rotational symmetry around poles p, q

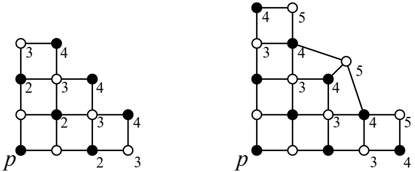
$B_k(p)$: the face-connected subsurface of G
(theorem: it really is one)
whose set of faces consists of all faces having at least
one of its vertices at distance at most $k-1$ from p



standard k-balls

G: a spherical quadrangulation having
order-n rotational symmetry around poles p, q

$B_k(p)$: the face-connected subsurface of G
(theorem: it really is one)
whose set of faces consists of all faces having at least
one of its vertices at distance at most $k-1$ from p



Proposition. if all vertices other than p in the interior of $B_k(p)$
have degree 4 in G,
then $B_k(p)$ is a standard k-ball of order degree(p)

standard k-balls

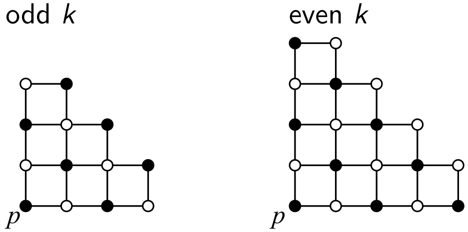
Proposition. if all vertices other than p in the interior of $B_k(p)$ have degree 4 in G, then $B_k(p)$ is a standard k-ball of order degree(p)

basic first step for understanding the structure of G:

grow $B_k(p)$ outwards from p
until $\partial B_k(p)$ hits an orbit of curvature vertices.

Curvature Vertices at Even k

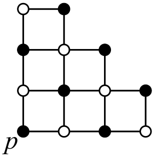
curvature vertices at even k



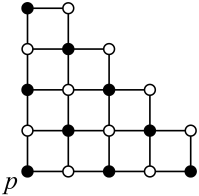
for k even, curvature vertices on $\partial B_k(p)$ lie on the "central ray" and can have only one orbit of curvature vertices on $\partial B_k(p)$

curvature vertices at even k

odd k

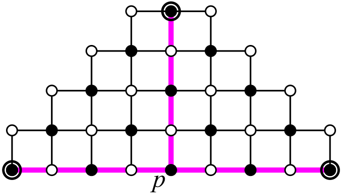


even k

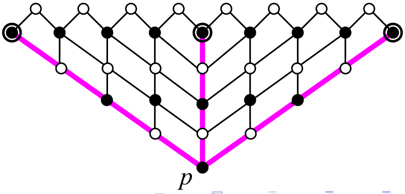


for k even, curvature vertices on $\partial B_k(p)$ lie on the “central ray” and can have only one orbit of curvature vertices on $\partial B_k(p)$

picture it this way



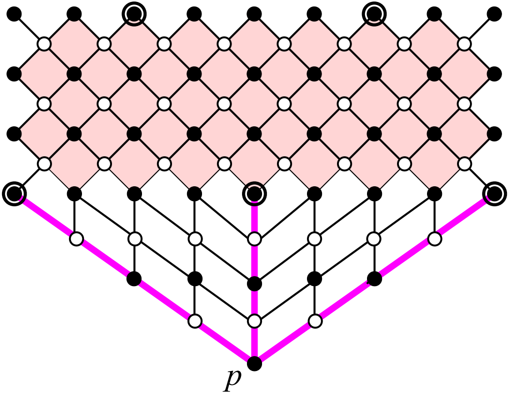
$=$



curvature vertices at even k

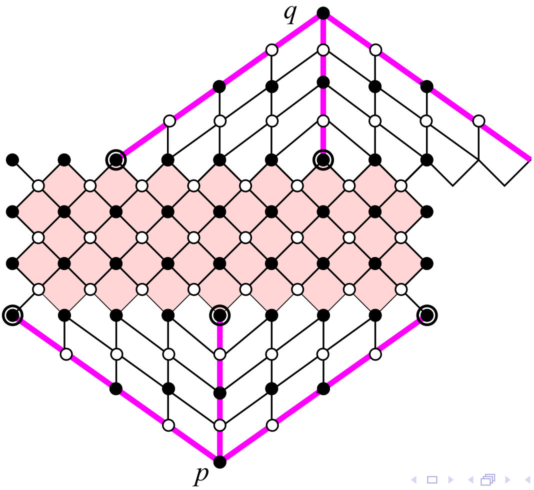
as t increases, $B_{k+t}(p)$ grows by adding flat layers
until $\partial B_{k+t}(p)$ hits the second orbit of degree-3 vertices

gaps within orbit are determined,
but overall offset from first orbit is not



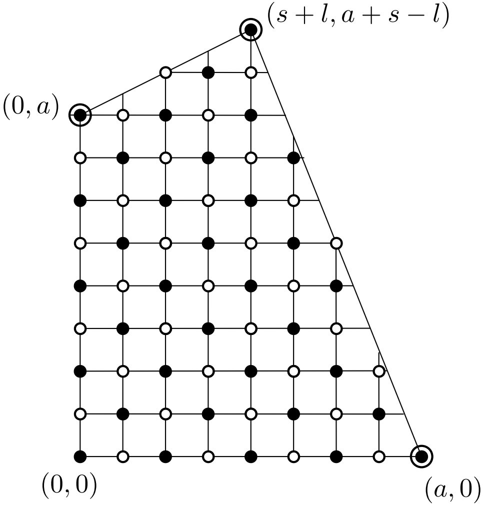
curvature vertices at even k

because $\partial B_k(p)$ is flat
and all degree-3 vertices have been encountered,
the only way to continue closes up the sphere with $B_k(q) \cong B_k(p)$

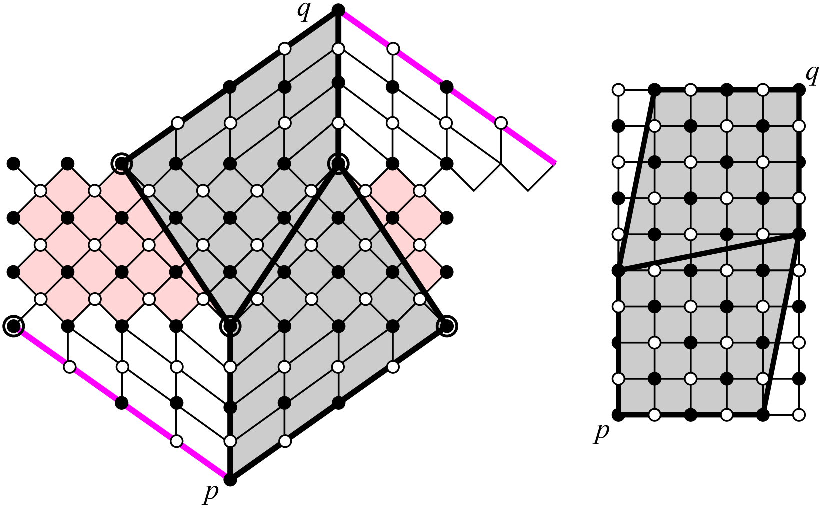


curvature vertices at even k

remember this panel?

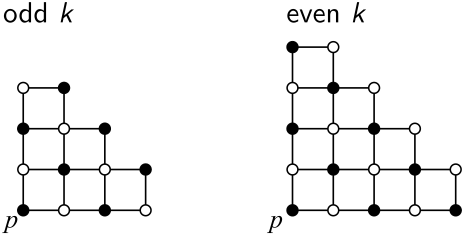


curvature vertices at even k



Curvature Vertices at Odd k

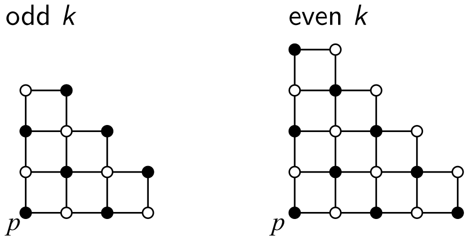
curvature vertices at odd k



for k odd

- curvature vertices on $\partial B_k(p)$ lie *off* the "central ray"

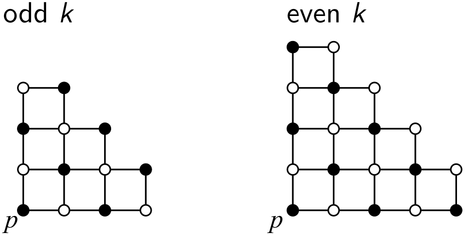
curvature vertices at odd k



for k odd

- curvature vertices on $\partial B_k(p)$ lie *off* the “central ray”
- can have two orbits of curvature vertices on $\partial B_k(p)$)

curvature vertices at odd k

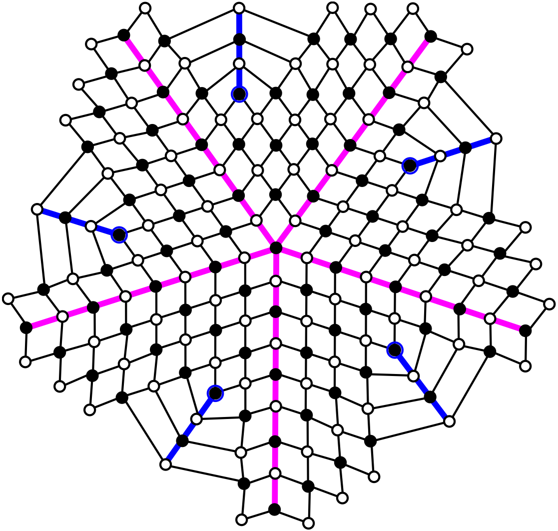


for k odd

- curvature vertices on $\partial B_k(p)$ lie *off* the "central ray"
- can have two orbits of curvature vertices on $\partial B_k(p)$)
- can have curvature vertices in the interior of $B_k(p)$ even when there are none in $\overline{B_{k-1}(p)}$)

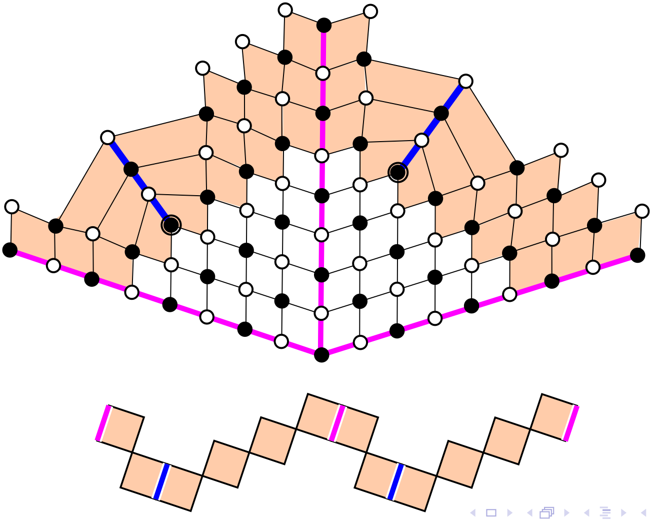
curvature vertices at odd k

with k odd, have "curvature rays" in addition to "central rays"



curvature vertices at odd k

as t increases, $B_{k+t}(p)$ grows by adding "zig-zag" layers until $\overline{B_{k+t}(p)}$ contains second orbit of degree-3 vertices (O_2)



curvature vertices at odd k

Note:
There are three options for the appearance of O_2 in $\overline{B_{k+t}(p)}$:

- If $k+t \equiv_2 0$ then the vertices of O_2 are the ends of the central rays from p.

curvature vertices at odd k

Note:
There are three options for the appearance of O_2 in $\overline{B_{k+t}(p)}$:

- If $k + t \equiv_2 0$ then the vertices of O_2 are the ends of the central rays from p.
- If $k + t \equiv_2 1$ then either

curvature vertices at odd k

Note:
There are three options for the appearance of O_2 in $\overline{B_{k+t}(p)}$:

- If $k + t \equiv_2 0$ then the vertices of O_2 are the ends of the central rays from p.
- If $k + t \equiv_2 1$ then either
  - the vertices of O_2 are the ends of the curvature rays from the first orbit of degree-3 vertices (O_1) and are in the interior of $B_{k+t}(p)$, or

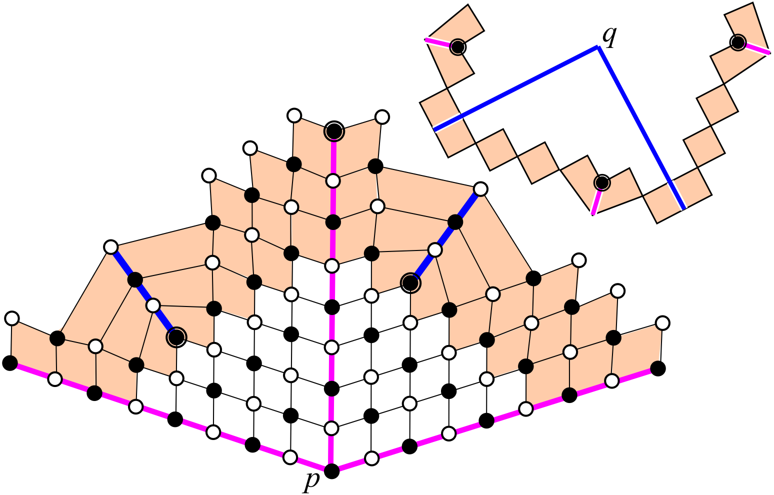
curvature vertices at odd k

Note:
There are three options for the appearance of O_2 in $\overline{B_{k+t}(p)}$:

- If $k + t \equiv_2 0$ then the vertices of O_2 are the ends of the central rays from p.
- If $k + t \equiv_2 1$ then either
  - the vertices of O_2 are the ends of the curvature rays from the first orbit of degree-3 vertices (O_1) and are in the interior of $B_{k+t}(p)$, or
  - the vertices of O_2 are on $\partial B_{k+t}(p)$ and are not the ends of the central rays from p nor the ends of the curvature rays from O_1.

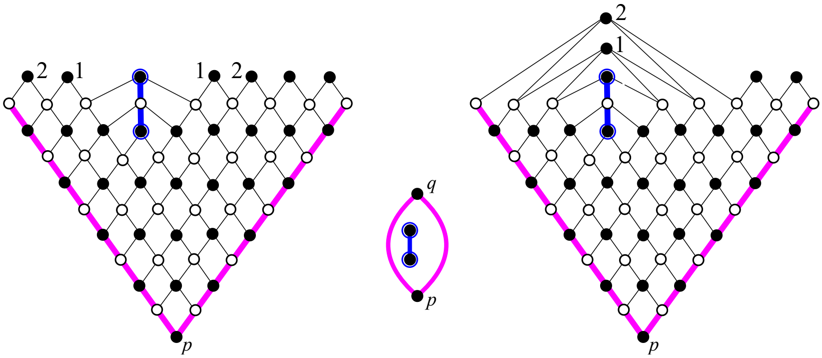
curvature vertices at odd k

If $k + t \equiv_2 0$ then the vertices of O_2 are the ends of the central rays from p.



curvature vertices at odd k

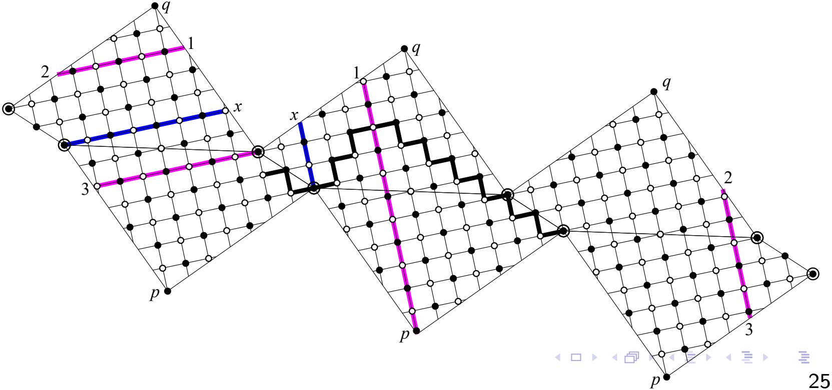
If $k + t \equiv_2 1$ then it can be that
the vertices of O_2 are the ends of the curvature rays
from the first orbit of degree-3 vertices (O_1)
and are in the interior of $B_{k+t}(p)$.



curvature vertices at odd k

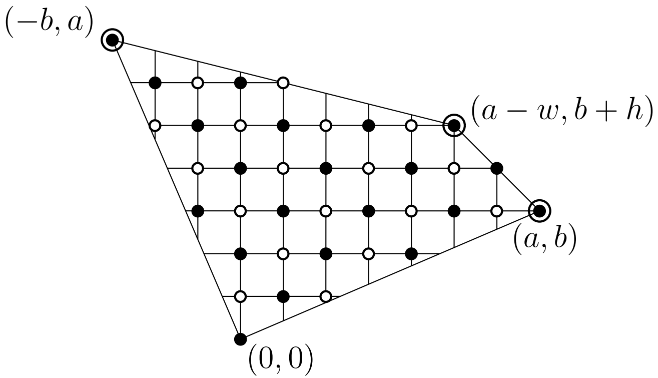
If $k + t \equiv_2 1$ then it can be that
the vertices of O_2 are on $\partial B_{k+t}(p)$
and are not the ends of the central rays from p
nor the ends of the curvature rays from O_1.

the central rays and the curvature rays "wander around,"
potentially crossing each other and themselves.



curvature vertices at odd k

remember this panel?

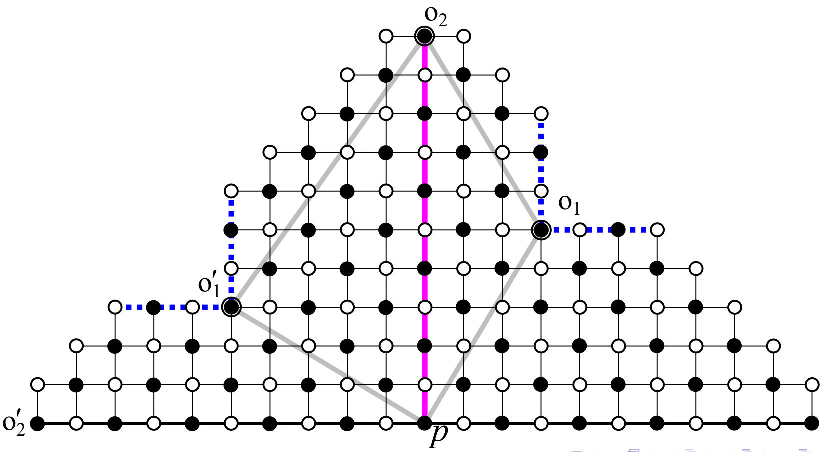


it is sufficiently generic to cover all cases when k is odd

curvature vertices at odd k

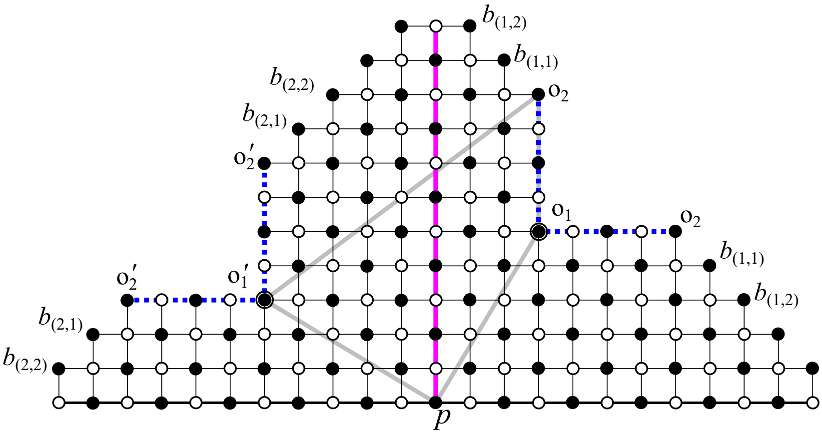
If $k + t \equiv_2 0$ then the vertices of O_2 are the ends of the central rays from p.

(cut the curvature rays so can embed in the $\{4, 4\}$ grid)



curvature vertices at odd k

If $k + t \equiv_2 1$ and the vertices of O_2
are the ends of the curvature rays from O_1



curvature vertices at odd k

If $k + t \equiv_2 1$ and the vertices of O_2 are on $\partial B_{k+t}(p)$ and are not the ends of the central rays from p nor the ends of the curvature rays from O_1.