Khovanov Homology and Categorification of Graph Polynomials

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Overview

We'll touch on the following topics:

- The Kauffman bracket and Jones polynomial
- Khovanov homology
- 2-dimensional TQFT's
- Generalized knot homology
- Graph homology and categorification of graph polynomials

Figures are taken from "Five Lectures on Khovanov Homology" by Paul Turner, unless otherwise specified.

The Kauffman Bracket and Jones Polynomial

The Kauffman bracket of a knot diagram D is defined recursively by smoothing crossings as follows:

$$\begin{aligned} \langle \times \rangle &= \langle \asymp \rangle - q \langle)\,(\rangle \\ \langle k \text{ circles in the plane } \rangle &= (q + q^{-1})^k \end{aligned}$$

This is not a knot invariant as it changes under $R1$ moves, but can be adjusted by a writhe factor to give the knot invariant

$$\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle.$$

This is the (unnormalized) Jones polynomial; $n_\pm$ denote the numbers of positive/negative crossings.)

To compute the Kauffman bracket, we smooth the crossings one at a time. Alternatively we can smooth all crossings simultaneously, which can be done in 2^n ways.

We say a smoothing with horizontal strands is a 0-smoothing and one with vertical strands is a 1-smoothing. This indexes the smoothings (or 'states') of a diagram by $\{0,1\}^n$. We have the following ‘state sum’ formula for $\hat{J}$:

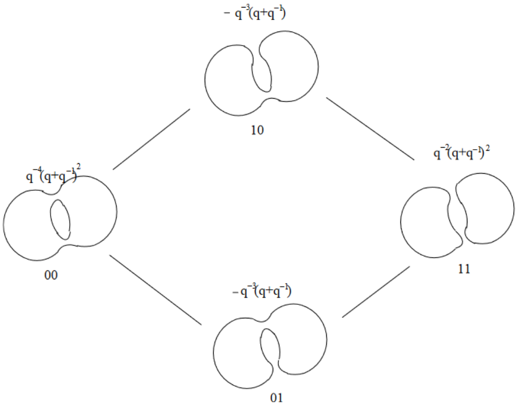
$$\hat{J}(D) = \sum_{\alpha \in \{0,1\}^n} (-1)^{r_\alpha + n_-} \; q^{r_\alpha + n_+ - 2n_-} \; (q + q^{-1})^{k_\alpha}.$$

Here r_α be the number of 1's in α and k_α is the number of circles in the smoothing specified by α.

The Kauffman Bracket and Jones Polynomial

As an example we compute $\hat{J}$ for the Hopf link:

$$
\begin{aligned}
\hat{J}(\bigcirc\!\!\!\!\bigcirc) &= q^{-4}(q+q^{-1})^2 - 2q^{-3}(q+q^{-1}) + q^{-2}(q+q^{-1})^2 \\
&= q^{-6} + q^{-4} + q^{-2} + 1.
\end{aligned}
$$



The Idea

Given a knot diagram D we assign to it, combinatorially, a bigraded chain complex and take homology:

$$D \mapsto C^{*,*}(D) \mapsto KH^{*,*}(D).$$

This bigraded homology is the **Khovanov homology** of D. We have:

- If D and D' are equivalent diagrams, $C^{*,*}(D)$ is homotopy equivalent to $C^{*,*}(D')$ and so $KH^{*,*}(D) \cong KH^{*,*}(D')$. Hence $KH^{*,*}(D)$ is a knot invariant.
- The graded Euler characteristic of $KH^{*,*}(D)$ is $\hat{J}(D)$:

$$\sum_{i \in \mathbb{Z}} (-1)^i \text{qdim}(KH^{i,j}(D)) = \hat{J}(D).$$

We say Khovanov homology **categorifies** the Jones polynomial.

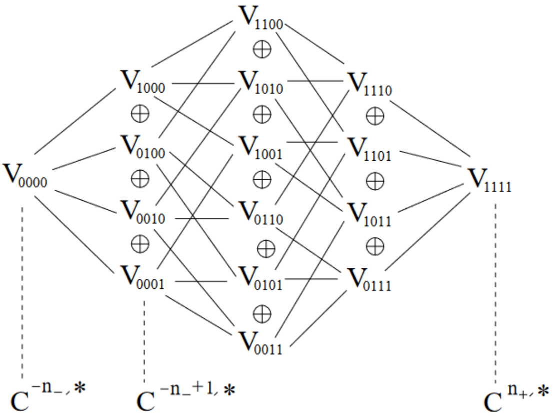
We let $V = \mathbb{Q}\{1, x\}$ be the vector space graded by $|1| = 1$ and $|x| = -1$. We consider the same smoothings given by $\alpha \in \{0,1\}^n$ and let

$$V_\alpha = V^{\otimes k_\alpha}\{r_\alpha + n_+ - 2n_-\}.$$

Here $\{\cdot\}$ denotes a shift in degree. Finally we define for each $i \in \mathbb{Z}$ and some knot diagram D:

$$C^{i,*}(D) = \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ r_\alpha = i + n_-}} V_\alpha.$$

The internal grading of each $C^{i,*}(D)$ arises because each V_α is graded. This internal grading is called the q-**grading**.

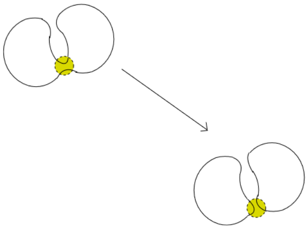


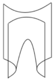
We have set things up so that if $v \in C^{*,*}(D)$ is (i,j)-graded, then

$$i = r_\alpha - n_- \qquad \text{and } j = |v| + i + n_+ - n_-.$$

Khovanov Homology

To take homology of $C^{*,*}(D)$ we need to turn it into a chain complex via a differential $d^i : C^{i,*}(D) \to C^{i+1,*}(D)$. Note this map turns one 0-smoothing into a 1-smoothing. This change looks like



We can think of such a change as a surface lying between the two circles, which is cylindrical everywhere except at the depicted disk, where we have put in a saddle .

Every change from a 0-smoothing to a 1-smoothing either merges two circles or splits them. Since we assign a copy of V to every circle we must thus build d^i out of maps $m : V \otimes V \to V$ and $\Delta : V \to V \otimes V$. We define

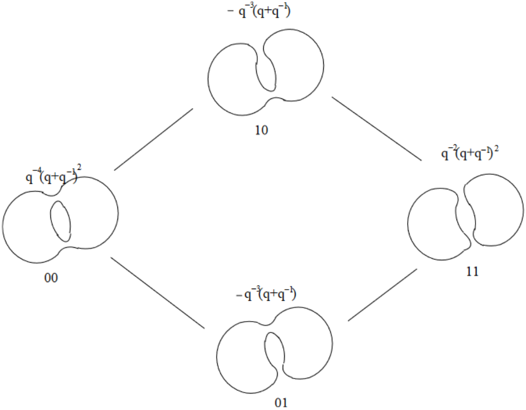
$$1^2 = 1, \quad 1x = x = x1, \quad x^2 = 0,$$
$$\Delta(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x.$$

This endows V with the structure of a **Frobenius algebra**. We use m and Δ to build d^i via

$$d^i(v) = \sum_{\zeta} \mathrm{sign}(\zeta) d_\zeta(v)$$

for $v \in V_\alpha \subseteq C^{i,*}(D)$. Here we sum over smoothing changes starting at α, and d_ζ is m or Δ depending on whether ζ splits or merges circles.

As an example we compute the Khovanov Homology of the Hopf link, recalling the following image used for the Jones polynomial:



Khovanov Homology

The Khovanov chain complex looks like

$$0 \longrightarrow C^{-2,*}(D) \xrightarrow{d} C^{-1,*}(D) \xrightarrow{d} C^{0,*}(D) \longrightarrow 0$$

where the nontrivial spaces and maps are given in more detail as

$$\begin{array}{ccccc} & & V\{-3\} & & \\ & \overset{m}{\nearrow} & & \overset{-\Delta}{\searrow} & \\ (V \otimes V)\{-4\} & & \oplus & & (V \otimes V)\{-2\} \\ & \underset{m}{\searrow} & & \underset{\Delta}{\nearrow} & \\ & & V\{-3\} & & \end{array}$$

Khovanov Homology

From this we can deduce the homology to be given by

Homological degree	*-2*	*-1*	*0*
Cycles	$\{1 \otimes x - x \otimes 1, x \otimes x\}$	$\{(1,1),(x,x)\}$	$\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$
Boundaries	-	$\{(1,1),(x,x)\}$	$\{1 \otimes x + x \otimes 1, x \otimes x\}$
Homology	$\{1 \otimes x - x \otimes 1, x \otimes x\}$	-	$\{1 \otimes 1, 1 \otimes x\}$
q-degrees	*-4, -6*		*0, -2*

The Khovanov homology is summarized in a table as

j \\ *i*	*-2*	*-1*	*0*
0			$\mathbb{Q}$
-1			
-2			$\mathbb{Q}$
-3			
-4	$\mathbb{Q}$		
-5			
-6	$\mathbb{Q}$		

We have now constructed a map that sends closed 1-manifolds (unions of circles) to vector spaces, and sends surfaces connecting them (smoothing changes) to linear maps. As such we have constructed a (symmetric, monoidal) functor

$$\mathtt{Bord}_2 \to \mathtt{Vect}_{\mathbb{Q}}.$$

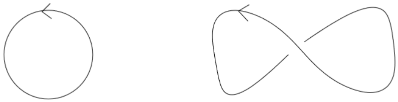
This is what we call a (closed) **2-dimensional topological quantum field theory**.

If you are familiar with TQFT's this is no surprise, since we know V is a commutative Frobenius algebra. In fact it turns out that there is a bijection

$$\{\text{TQFT} : \texttt{Bord}_2 \to \texttt{Vect}_{\mathbb{Q}}\} \leftrightarrow \{\text{Comm. Frob. alg. over } \mathbb{Q}\}$$

It is therefore natural to ask: can we replace V by some other TQFT/Frobenius algebra A and still obtain a knot homology theory?

The fact that we are looking for knot invariant means that we are subject to some constraints imposed by the equivalence relation on knot diagrams. For example consider the following diagrams:



They give rise to chain complexes

$$0 \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow A \otimes A \longrightarrow 0$$

These must have equal Euler characteristic, so

$$\dim(A) = -\dim(A) + \dim(A \otimes A) \implies \dim(A) = 2.$$

It turns out the restrictions imply that $A = \mathbb{Q}\{1, x\}$ as before, with Frobenius algebra structure $A_{h,t}$ indexed by $(h, t) \in \mathbb{Q}^2$:

$$1^2 = 1, \quad 1x = x = x1, \quad x^2 = hx + t1,$$
$$\Delta(1) = 1 \otimes x + x \otimes 1 - h1 \otimes 1, \quad \Delta(x) = x \otimes x + t1 \otimes 1.$$

Khovanov homology corresponds to $(h, t) = (0, 0)$. Taking $(h, t) = (0, 1)$ gives **Lee homology**. In fact up to isomorphism these are the only two rational knot homology theories.

Side-track: Rasmussen's s-invariant

Lee theory contains an invariant called Rasmussen's s-invariant. Interestingly it can be used to construct an exotic structure on $\mathbb{R}^4$ as follows:

- Given a knot K we construct X_K by gluing $D^2 \times D^2$ to D^4 along K.
- X_K embeds into $\mathbb{R}^4$ topologically resp. smoothly iff K is topologically slice resp. smoothly slice.
- The smooth structure of X_K extends along an embedding $X_K \hookrightarrow \mathbb{R}^4$ to a smooth structure on $\mathbb{R}^4$.
- K is topologically slice if $\Delta_K(t) = 1$, and smoothly slice only if its s-invariant is 0.
- So it suffices to find K with $\Delta_K = 1$ and $s \neq 0$. An example is the $(-3, 5, 7)$ pretzel knot, which has 15 crossings.

An exotic smooth structure on $\mathbb{R}^n$ exists only for $n = 4$.

Graph Homology

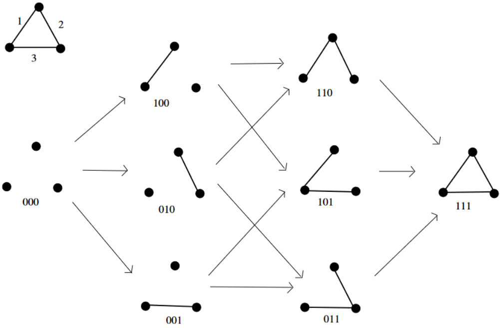
These techniques extend in a straight-forward manner to give categorifications of graph polynomials. That is, graph homology theories whose graded Euler characteristics are graph polynomials. In fact, these homology theories are **easier** to construct!

As an example, let's categorify the chromatic polynomial. First we need a state sum. For $G = (V, E)$ and $\alpha \in \{0, 1\}^{|E|}$ we let $G_\alpha = (V, \{e \in E : \alpha(e) = 1\}) := (V, E_\alpha)$. Then

$$P(G) = \sum_{\alpha \in \{0,1\}^{|E|}} (-1)^{r_\alpha} \lambda^{k_\alpha},$$

where $r_\alpha = |E_\alpha|$ and k_α is the number of components in G_α.

We now take **any** graded algebra R (doesn't need to be a 2D Frobenius algebra), and for $\alpha \in \{0,1\}^{|E|}$ we let $R_\alpha = R^{\otimes k_\alpha}$. We form an n-hypercube as before:



Graph Homology

Finally for the chain complex we set

$$C^{i,*} = \bigoplus_{r_\alpha = i} R_\alpha$$

with differential defined for $v \in R_\alpha \subseteq C^{i,*}$ by

$$d^i(v) = \sum_{\zeta} \mathsf{sign}(\zeta) d_\zeta(v),$$

where d_ζ is the identity or the multiplication on R, depending on whether ζ merges two components or not.

Graph Homology

Taking the homology of $C^{i,*}(G)$ we obtain $H^{*,*}(G)$ which categorifies the chromatic polynomial in the following sense:

$$\sum_{i,j}(-1)^i q^j \dim(H^{i,j}(G)) = P(G)|_{\lambda=\mathsf{qdim}(R)}$$

Here $\mathsf{qdim}(R)$ is the **graded dimension** of the graded algebra $R = \bigoplus_{m\in\mathbb{N}} R^m$, given by

$$\mathsf{qdim}(R) = \sum_{m\in\mathbb{N}} q^m R^m.$$

The chromatic graph homology also gives rise to a categorification of the deletion-contraction relation $P(G) = P(G-e) - P(G/e)$. Namely for all (i,j) there is a short exact sequence:

$$0 \longrightarrow C^{i,j}(G/e) \longrightarrow C^{i,j}(G) \longrightarrow C^{i,j}(G-e) \longrightarrow 0$$

Summing over j and applying the snake lemma we obtain a long exact sequence

$$0 \longrightarrow H^{0,*}(G) \longrightarrow H^{0,*}(G-e) \longrightarrow H^{0,*}(G/e) \xrightarrow{\ d^*\ } H^{1,*}(G) \longrightarrow H^{1,*}(G-e) \longrightarrow \dots$$

Results on Graph Homology

Work so far on graph homology is limited: below is a series of 5 papers, written by Yongwu Rong et al.

- "A categorification for the chromatic polynomial" (0412264)
- "Graph Cohomologies from Arbitrary Algebras" (0506023)
- "Torsion in Graph Homology" (0507245)
- "Knight move for chromatic graph cohomology " (0511598)
- "A categorification for the Tutte polynomial" (0512613)

Open research directions

Some avenues for further investigation:

- Categorification of matching polynomial?
- Computer implementation?
- Can we extend the relation between Jones and Tutte polynomials to homology?
- Can we categorify specialization of the Tutte polynomial?
- Torsion in graph homology.

References

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- Turner, Paul. "Five lectures on Khovanov homology." arXiv preprint math/0606464 (2006).
- Bar-Natan, Dror. "On Khovanov's categorification of the Jones polynomial." Algebraic Geometric Topology 2.1 (2002): 337-370.
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